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Breather solutions and propagation behaviors for the (3+1)-dimensional generalized shallow water equation

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Abstract: Some breather wave solutions, which include breather solitary wave solutions, breather lump wave solutions, are obtained for the (3+1)-dimensional generalized shallow water equation through the Hirota bilinear method and test function method. Propagation behaviors of three breather waves are analyzed through figures. These results reveal more nonlinear phenomena for the (3+1)-dimensional generalized shallow water equation.

Keywords: The shallow water equation, the Hirota bilinear method, breather wave solution, propagation behavior.

I. Introduction

The (3+1)-dimensional generalized shallow water equation

$$u_{xxxy} - 3u_x u_{xy} - 3u_y u_{xx} + u_{xt} - u_{xz} = 0, \quad (1)$$

where describes the propagation of long water waves in the ocean, estuaries and reservoirs, and has applications in weather simulations, tidal waves, river and irrigation flows, and tsunami prediction and so on[1,2]. Many researchers have studied this equation and obtained innovative results which include the structures and propagation behavior of the solutions [1-13], such as multiple-soliton solutions[1], the lump solutions[2] , traveling wave solutions and non-traveling wave solutions[3], rational solutions and lump solutions[4,5],periodic solitary wave solutions[6], Grammian and Pfaffian solutions[7]. Some researchers have also studied the variable-coefficient (3+1)-dimensional generalized shallow water wave equation and have obtained results similar to the previous ones[8,9]. In the previously obtained lump solutions, they are limited to the special situation for $z = x$ or $y = x$. Among the results obtained, the Hirota bilinear method is the most applied method[1-4,7-9,11]. Although solutions of various structures have been obtained for the (3+1)-dimensional generalized shallow water equation, however, I don't think the breather solutions are fully presented, such as breather solitary wave solutions, breather lump solutions, etc..

In this paper，we continue to apply the Hirota bilinear method to construct breather-like solutions which include breather solitary wave solutions and breather lump solutions, and analyze the behavior of breather waves by using figures.

Rest of the paper is organized as follows: In Sect.2, we will derive the bilinear forms of Eq.(1) and solve bilinear equation by the test functions. In Sect.3, analyzing the behaviors of breather waves by figures. Sect.4 will be our conclusions.

II. Breather solutions of Eq.(1)

By the transformation

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$$u(x,y,z,t) = -2(\ln f)_x, \tag{2}$$

Eq.(1) can be changed into the following bilinear equation

$$f_{yt} f - f_y f_t + f_{xxxy} f - f_{xxx} f_y - 3 f_{xxy} f_x + 3 f_{xx} f_{xy} - f_{xz} f + f_x f_z = 0, \tag{3}$$

2.1 Breather solitary waves

First of all, we construct breather solitary wave solutions and assume that[14]

$$f(x,y,z,t) = G + Ae^{(K\cdot x + L\cdot y + M\cdot z + N\cdot t)} + Be^{-(K\cdot x + L\cdot y + M\cdot z + N\cdot t)} + C\cos(Px + Qy + Rz + St), \tag{4}$$

where A, B, C, G, K, L, M, N, P, Q, R and S are real constants to be determined. We substitute Eq.(4) into Eq.(3), and collect the coefficients of $e^{i\cdot(K\cdot x+L\cdot y+M\cdot z+N\cdot t)}$ (i = 0,1,2), $\cos(Px+Qy+Rz+St)$ and $\sin(Px+Qy+Rz+St)$ to get the system of equations about A, B, C, G, K, L, M, N, P, Q, R and S (the tedious calculation process is omitted). Solving these equations, we obtain the following results.

Case1:

$$f_1(x,y,z,t) = G + \frac{CQ^2}{L^2}\cosh\left(\frac{PL}{Q}x - Ly - \frac{L(3Q^2P^2 + 2QR + 3P^2L^2)}{2Q^2}z + \frac{PL(3Q^2P^2 + 2QR + P^2L^2)}{2Q^3}t\right)$$
$$+ C\cos\left(Px + Qy + Rz + \frac{P(R+P^2Q)}{Q}t\right), \tag{5}$$

where C, G, L, P, Q and R are arbitrary real constants.

Substituting Eq.(5) into Eq.(2), we can obtain solutions of Eq.(1) as follows

$$u_1(x,y,x,t) = -2\frac{\frac{PCQ}{L}\sinh(\psi) - CP\sin\left(Px + Qy + Rz + \frac{P(R+P^2Q)}{Q}t\right)}{G + \frac{CQ^2}{L^2}\cosh(\psi) + C\cos\left(Px + Qy + Rz + \frac{P(R+P^2Q)}{Q}t\right)}, \tag{6}$$

where $\psi = \frac{PL}{Q}x - Ly - \frac{L(3Q^2P^2 + 2QR + 3P^2L^2)}{2Q^2}z + \frac{PL(3Q^2P^2 + 2QR + P^2L^2)}{2Q^3}t$. Due to the function of the triangular periodic function, the solitary wave can produce breathing effects in each direction of space, thus this solution is called a breather solitary wave solution.

Case2:

$$f_2(x,y,z,t) = G + 2A\cosh(Ly + Mz) + C\cos\left(Px + \frac{P(P^2L + M)}{L}t\right), \tag{7}$$

where A,C,G,L,M and P are arbitrary real constants. Substituting Eq.(7) into Eq.(2), another solution of Eq.(1) can be written as

$$u_2(x,y,z,t)=\frac{2CP\sin(Px+\frac{P(P^2L+M)}{L}t)}{G+2A\cosh(Ly+Mz)+C\cos(Px+\frac{P(P^2L+M)}{L}t)}, \tag{8}$$

Similarly, this solitary wave produces a breathing effect in the x-axis direction.

Case3:

$$f_3(x,y,z,t)=G+2A\cosh(Kx+Nt)+C\cos(Qy+\frac{Q(K^3+N)}{K}z), \tag{9}$$

where A,C,G,K,N and Q are arbitrary real constants. Substituting Eq.(9) into Eq.(2), the third solution of Eq.(1) can be written as

$$u_3(x,y,z,t)=-\frac{4AK\sinh(Kx+Nt)}{G+2A\cosh(Kx+Nt)+C\cos(Qy+\frac{Q(K^3+N)}{K}z)}, \tag{10}$$

2.2 Lump and breather lump waves

Secondly, we consider the lump solutions and breather lump solutions. Test functions are represented as two structures[5,14].

Case1:

$$f(x,y,z,t)=A(K\cdot x+L\cdot y+M\cdot z+N\cdot t)^2+B(P\cdot x+Q\cdot y+R\cdot z+S\cdot t)^2+C, \tag{11}$$

Substituting Eq.(11) into Eq.(3), f(x,y,z,t) can be obtained which is expressed as

$$\begin{aligned} f_4(x,y,z,t)=&A(K\cdot x+L\cdot y+M\cdot z+\frac{K(MC-3BP^2L-3ALK^2)}{LC}t)^2\\ &+B(P\cdot x+\frac{3AKL(BP^2+AK^2)}{BCP}z+\frac{(3ABP^2LK^2+3A^2K^4L+BCP^3M)}{BCPL}t)^2+C, \end{aligned} \tag{12}$$

and

$$\begin{aligned} f_5(x,y,z,t)=&\frac{3BPQ(BP^2+AK^2)}{AMK}+A(K\cdot x+M\cdot z+\frac{(RK+MP)}{Q}t)^2\\ &+B(P\cdot x+Q\cdot y+R\cdot z+\frac{(BPR-AMK)}{BQ}t)^2, \end{aligned} \tag{13}$$

where A, B, C, K, L, M, P, Q and R are real constants. Accordingly, the lump solutions of Eq.(1) can be obtained through Eq.(2)

$$u_4(x,y,z,t)=-2\ln(f_4(x,y,z,t))_x\ ,\ \ u_5(x,y,z,t)=-2\ln(f_5(x,y,z,t))_x. \tag{14}$$

Case2:

$$f(x,y,z,t)=A(K\cdot x+L\cdot y+M\cdot z+N\cdot t)^2+B\cos(P\cdot x+Q\cdot y+R\cdot z+S\cdot t)+C, \tag{15}$$

Similarly, the expression for $f(x,y,z,t)$ can be found and expressed as

$$f_6(x,y,z,t)=\frac{BP^2}{2K^2}(K\cdot x+L\cdot y+\frac{L(P^3+2S)\cdot}{2P}z+\frac{K(P^3+2S)}{2P}t)^2 \\ +B\cos(P\cdot x-\frac{PL}{K}y-\frac{L(S-P^3)}{K}z+S\cdot t)+C \quad , \tag{16}$$

and

$$f_7(x,y,z,t)=\frac{1}{2}B(P\cdot x-Q\cdot y)^2+B\cos(P\cdot x+Q\cdot y-\frac{3}{2}QP^2\cdot z-\frac{1}{2}P^3\cdot t)+C\,, \tag{17}$$

where B,C,K,L,P,Q and S are real constants. The lump breather solutions of Eq.(1) can be written as

$$u_6(x,y,z,t)=-2\ln(f_6(x,y,z,t))_x\ ,\ u_7(x,y,z,t)=-2\ln(f_7(x,y,z,t))_x. \tag{18}$$

III. The propagation behaviors of breather waves

From $u_1(x,y,x,t)=-2\dfrac{\frac{PCQ}{L}\sinh(\psi)-CP\sin(Px+Qy+Rz+\frac{P(R+P^2Q)}{Q}t)}{G+\frac{CQ^2}{L^2}\cosh(\psi)+C\cos(Px+Qy+Rz+\frac{P(R+P^2Q)}{Q}t)}$,

$\psi=\dfrac{PL}{Q}x-Ly-\dfrac{L(3Q^2P^2+2QR+3P^2L^2)}{2Q^2}z+\dfrac{PL(3Q^2P^2+2QR+P^2L^2)}{2Q^3}t$ and Fig.1, we find that by properly selecting the parameters, one can obtain breather waves in different directions. In Fig.1, there are a shock wave along the X direction and breather waves along the Y direction, forming a waterfall structure.

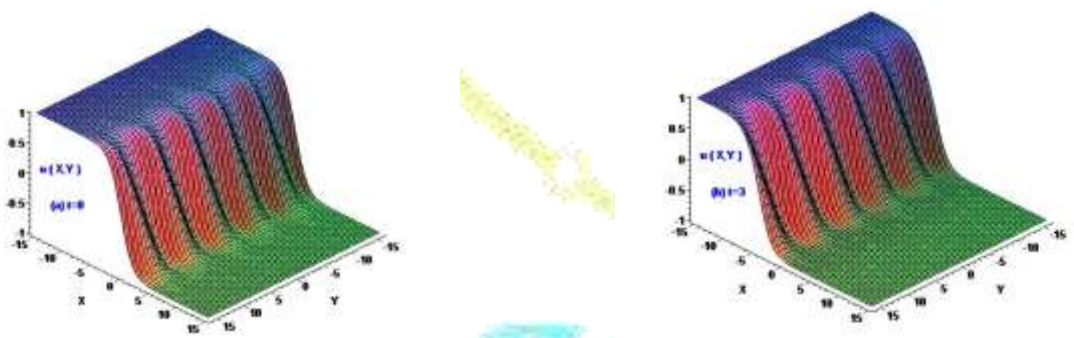

Fig. 1: The breather solitary wave solution u 1 (x,y,x,t) with C = 3, G = 6, P =12 , Q = 1, L = 1, R = 3 and $X=\frac{1}{2}x-y-\frac{15}{4}z,\quad Y=\frac{1}{2}x+y+3z$ when t = 0 and t = 3.

As far as $u_2(x,y,z,t)=\dfrac{2CP\sin(Px+\frac{P(P^2L+M)}{L}t)}{G+2A\cosh(Ly+Mz)+C\cos(Px+\frac{P(P^2L+M)}{L}t)}$ is concerned, a single solitary wave forms a breathing effect in the x direction, and then form a continuous solitary wave chain (see Fig.2). This structure has physical application value, such as the transmission of electronic or optical signals.

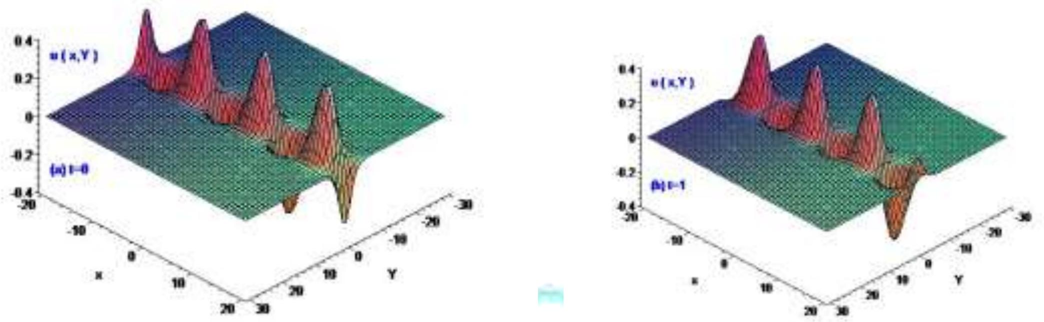

Fig. 2: The breather solitary wave solution $u_2(x,y,z,t)$ with A = 1, C = 3, G = 6, L = $\frac{1}{3}$, M = 2, P = $\frac{1}{2}$, Q = 1, R = 3 and $Y=\frac{1}{3}y+2z$ when t = 0 and t = 1.

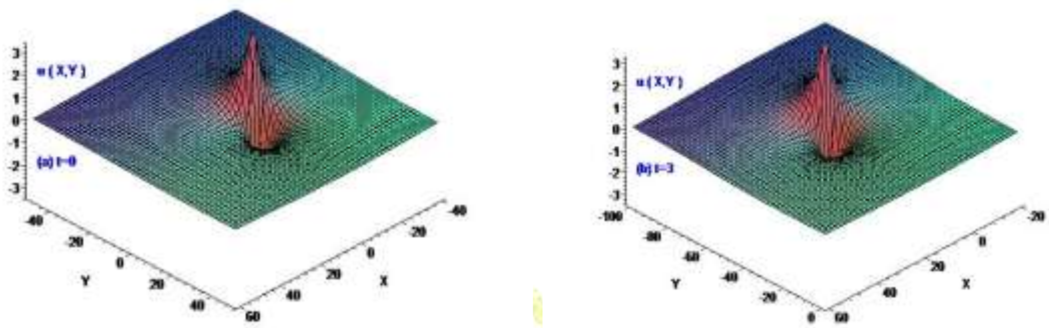

Fig. 3: The lump solution $u_4(x,y,z,t)$ with A = 2, B = 2, C = 6, K = 1L = 1, M = 5, P = 3 and $X=x+y+5z$, $Y=3x+\frac{10}{3}z$ when t = 0 and t = 3.

Fig.3 is profiles of lump solution $u_4(x,y,z,t)$ when t = 0 and t = 3. From the time and coordinate changes, we clearly see the movement of this wave. After this wave is superimposed with a periodic wave, a lump breather wave is formed (see Fig.4). This breather waves are sometimes severely destructive, for example, if water waves at sea form this breathing effect, they will cause continuous impact damage to obstacles. However, in practical applications, if people want to enhance signals, they can also be generated using this structure.

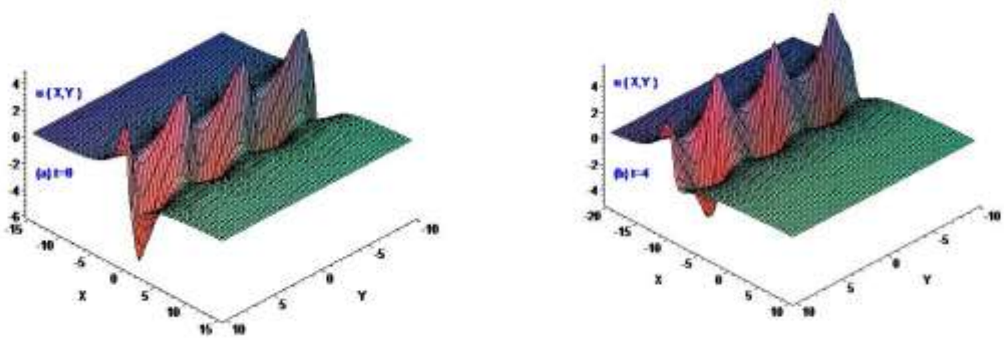

Fig. 4: The breather lump solution $u_6(x,y,z,t)$ with B = 4, C = 6, K = 1, L = 2, P = 2, S = 1 and

$X = x + 2y + 5z$, $Y = 2x - 4y + 14z$ when t = 0 and t = 4.

IV. Conclusion

The (3+1)-dimensional generalized shallow water equation is similar to many other non-linear equations with breather waves, such as breather solitary waves, breather lump waves. Some breather waves have application value, and some breather waves are destructive. Revealing these nonlinear phenomena has physical application significance. One question deserves further consideration: how to multi-dimensionally characterize the propagation of multi-dimensional waves?

Acknowledgments

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