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Module approximate amenability of $l^1(S)$ and 2-weak module amenability of Beurling semigroup algebras

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Abstract. In this paper, we study the module approximate amenability of semigroup algebra $l^1(S)$ as $l^1(E)$-module, where S is an inverse semigroup and E is the set of idempotents of S. Also, we show that the weighted algebra $l^1(S,\omega)$ as $l^1(E,\omega)$-module is 2-weakly module amenable.

1. Introduction

The concept of amenability of Banach algebras was first introduced by B. E. Johnson in [**4**]. M. Amini in [**1**] introduced the notion of module amenability for a class of Banach algebras which could be considered as a generalization of the Johnson's amenability. He showed that for an inverse semigroup S with the set of idempotents E, the semigroup algebra $l^1(S)$ on $l^1(E)$ is module amenable if and only if S is amenable.

The concept of approximately amenable Banach algebras was intiated by Ghahramani and Loy in [**18**]. They showed that the group algebra $L^1(G)$ is approximately amenable if and only if G is amenable where G is locally compact. This is fails to be true for discrete semigroup. If S is discrete semigroup, then approximate amenability of $l^1(S)$ implies amenability of S.

Recently Bami and Samea have shown the above result for the case that S is cancellative semigroup. Aghababa and Bodaghi in [**19**] defines the notions of module approximate amenability and module approximate contractibility for a Banach algebra $\mathcal{A}$ which is also a Banach $\mathfrak{A}$-module with compatible actions that introduced in[**1**]. They showed if S is an inverse semigroup with the set of idempotents E, then the semigroup algebra $l^1(S)$ is $l^1(E)$-

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module approximately amenable if and only if S is amenable, where $l^1(E)$ acts on $l^1(S)$ with trivial left action.

The concept of weak amenability was first introduced by Bade, Curtis and Dales in [**2**] for commutative Banach algebras. B. E. Johnson in [**7**] extended to the non-commutative case.

Let $\mathcal{A}$ be a Banach algebra and $n \geq 0$ be an integer. A Banach algebra $\mathcal{A}^{(n)}$ be the $n-$th dual module of $\mathcal{A}$ when $n > 0$ and be $\mathcal{A}$ itself when $n = 0$. A Banach algebra $\mathcal{A}$ is called *weakly amenable* [**2**] if every bounded derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is inner. Also a Banach algebra $\mathcal{A}$ is called *n-weakly amenable* [**?**] if every bounded derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^{(n)}$ are inner. The Banach algebra $\mathcal{A}$ is *permanently weakly amenable* [**?**] if it is *n-weakly amenable* for all $n \geq 1$.

Let $L^1(G,\omega)$ be a Beurling algebra on a locally compact abelian group G. The case of weak amenability has been studied in [**2**] and [**5**]. One major result states that $L^1(Z,\omega)$ is weakly amenable if and only if $inf_n \frac{\omega(n)\omega(-n)}{n} = 0$. Therefore $l^1(G,\omega)$ is weakly amenable if $inf_n \frac{\omega(nt)\omega(-nt)}{n} = 0$ for all $t \in G$.

Dales and Lau in [**16**] showed that if $\omega \geq 1$ and $inf_n \frac{\omega(nt)}{n} = 0$ for all $t \in G$, then $L^1(G,\omega)$ is *2-weakly amenable*.

In this paper, we consider the canonical actions of $l^1(E)$ on $l^1(S)$ and discuss the module approximate amenability of $l^1(S)$. we investigate the *2-weak module amenability* for $l^1(S,\omega)$ where S is an inverse semigroup and ω be a weight on S.

2. Module approximate amenability of semigroup algebras with new actions

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras and let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module such that

$$\alpha.(ab) = (\alpha.a).b \qquad (ab).\alpha = a(b.\alpha) \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

If X is both a Banach $\mathcal{A}$-module and a Banach $\mathfrak{A}$-module such that for all $a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}$

$$\alpha.(a.x) = (\alpha.a).x \quad (a.x).\alpha = a.(x.\alpha) \quad x.(a.\alpha) = (x.a).\alpha \quad x.(\alpha.a) = (x.\alpha).a,$$

then X is called an $\mathcal{A}$-$\mathfrak{A}$-*module*. If moreover,

$$\alpha.x = x.\alpha \qquad (\alpha \in \mathfrak{A}, x \in X),$$

then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module.

Let X and Y be $\mathcal{A}$-$\mathfrak{A}$-modules and let $\phi : X \to Y$ be a linear map which satisfies the following conditions:

$$\phi(\alpha.x) = \alpha.\phi(x) \qquad \phi(x.\alpha) = \phi(x).\alpha$$
$$\phi(a.x) = a.\phi(x) \qquad \phi(x.a) = \phi(x).a \qquad (a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}).$$

Then ϕ is called an $\mathcal{A}$-$\mathfrak{A}$-*module bi homomorphism.* Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then the projective tensor product $\mathcal{A}\hat{\otimes}X$ is also an $\mathcal{A}$-$\mathfrak{A}$-module with the following actions:

$$a.(b\otimes x) = (ab)\otimes x \qquad (b\otimes x).a = b\otimes(x.a)$$
$$\alpha.(b\otimes x) = (\alpha.b)\otimes x \qquad (b\otimes x).\alpha = b\otimes(x.\alpha) \qquad (a,b\in\mathcal{A}, x\in X, \alpha\in\mathfrak{A}).$$

Now, let $\pi:\mathcal{A}\hat{\otimes}X\to X$ be defined by

$$\pi(a\otimes x) = a.x \qquad (a\in\mathcal{A}, x\in X).$$

It follows from the definition that π is an $\mathcal{A}$-$\mathfrak{A}$-module bi homomorphism.

Let I_X be the closed $\mathcal{A}$-$\mathfrak{A}$-submodule of the projective tensor product $A\hat{\otimes}X$ generated by

$$\{(a.\alpha)\otimes x - a\otimes(\alpha.x) : a\in\mathcal{A}, \alpha\in\mathfrak{A}, x\in X\}.$$

Let J_X be the closed submodule of X generated by $\pi(I_X)$. That is

$$J_X = \overline{\langle\pi(I_X)\rangle}.$$

In special case, when $X=\mathcal{A}$, $J_{\mathcal{A}}$ is the closed ideal generated by $\{(a.\alpha)b - a(\alpha.b) : a,b\in\mathcal{A}, \alpha\in\mathfrak{A}\}$ and the quotient Banach algebra $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is also an $\mathcal{A}$-$\mathfrak{A}$-module.

Lemma 2.1. *Let X^* be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and let $D:\mathcal{A}\to X^*$ be a module derivation. Then $D(\mathcal{A})\subseteq J_X^{\perp}$.*

Proof. Let $a,b\in\mathcal{A}, \alpha\in\mathfrak{A}$ *and* $x\in X$, then $(a.\alpha)x - a(\alpha.x)\in J_X$. Then

$$\langle D(b), (a.\alpha)x - a(\alpha.x)\rangle = \langle D(b)(a.\alpha) - (D(b)a).\alpha, x\rangle = 0.$$

□

For Banach algebras $\mathcal{A}$, $\mathfrak{A}$ and a Banach $\mathcal{A}$-$\mathfrak{A}$-module X with compatible actions, a bounded map $D:\mathcal{A}\to X$ is called a *module derivation* [**1**] if D satisfies the following:

$$D(ab) = D(a).b + a.D(b)$$
$$D(\alpha.a) = \alpha.D(a), \qquad D(a.\alpha) = D(a).\alpha \qquad (a,b\in\mathcal{A}, \alpha\in\mathfrak{A}).$$

Note that $D:\mathcal{A}\to X$ is bounded if there exists $M>0$ such that for every $a\in\mathcal{A}$, $\|D(a)\|\le M\|a\|$. If X is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, then for each $x\in X$ defines an module derivation as $D_x(a) = a.x - x.a$, $(a\in\mathcal{A})$. These are called inner module drivation.

Definition 2.2. A Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$- module) if for every Banach $\mathcal{A}$-$\mathfrak{A}$-module X^* with commutative $J_X^{\perp}$ (as an $\mathfrak{A}$- module) and $a.(\alpha.y) = (a.\alpha).y \quad (a\in\mathcal{A}, \alpha\in\mathfrak{A}, y\in J_X^{\perp})$, each module derivation $D:\mathcal{A}\to J_X^{\perp}$ is inner .

Note that if $\mathfrak{A} = \mathbb{C}$, then the module amenability will absolutely overlap with Johnson's amenability [**4**] for a Banach algebra.

Let X is a commutative $\mathcal{A}$-$\mathfrak{A}$-module. For each net $x_\alpha \in X$ defines an approximate inner module derivation as

$$D_{x_\alpha}(a) = lim_\alpha(a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in \mathcal{A}).$$

Definition 2.3. The Banach algebra $\mathcal{A}$ is called *module approximate amenable* (as an $\mathfrak{A}$-module) if for every Banach $\mathcal{A}$-$\mathfrak{A}$-module X^* with commutative $J_X^\perp$ (as an $\mathfrak{A}$- module) and $a.(\alpha.y) = (a.\alpha).y \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, y \in J_X^\perp)$, each $\mathfrak{A}$-module derivation $D : \mathcal{A} \longrightarrow J_X^\perp$ is approximate inner.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with commutative canonical actions. If J_0 be a closed ideal of $\mathcal{A}$ such that $J_\mathcal{A} \subseteq J_0$, then $\frac{\mathcal{A}}{J_0}$ is commutative Banach $\mathfrak{A}$-module.*

Proof. If canonical actions are commutative, then $J_\mathcal{A} = \{0\}$. Therefore $0 = a.\alpha - \alpha.a \in J_0$. By [**17**, Theorem–], $\frac{\mathcal{A}}{J_0}$ is commutative banach $\mathfrak{A}$-module . □

Theorem 2.5. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with caconical actions and J_0 be a closed ideal of $\mathcal{A}$ such that $J_\mathcal{A} \subseteq J_0$, then for any commutative Banach $\mathfrak{A}$-module $J_X^\perp$ with canonical actions every module derivation $D : \frac{\mathcal{A}}{J_0} \longrightarrow J_X^\perp$ is approximate inner. So $\frac{\mathcal{A}}{J_0}$ is module approximate amenable*

Proof. Suppose X be a unital commutative $\frac{\mathcal{A}}{J_0}$-$\mathfrak{A}$- bimodule and $D : \frac{\mathcal{A}}{J_0} \longrightarrow J_X^\perp$ be a bounded module derivation. Let $a.x = (a + J_0).x$ and $x.a = x.(a + J_0) \quad (a \in \mathcal{A}, x \in X)$. In this case $\mathcal{A}$ becomes to commutative Banach $\mathcal{A}$-bimodule . In the other hand we define

$$\alpha.x = \alpha * x, \quad x.\alpha = x * \alpha \quad (x \in X, \alpha \in \mathfrak{A}).$$

Thus X is a Banach $\mathfrak{A}$-module. If we suppose $x.\alpha = \alpha.x$ then X is commutative Banach $\mathcal{A} - \mathfrak{A}$- module . Conside r $\overline{D} : \mathcal{A} \longrightarrow X^*$ with $\overline{D}(a) = D(a + J_0)$ and show that $\overline{D}$ is a module derivation. Let $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$ we have

$$\overline{D}(a \pm b) = D(a \pm b + J_0) = D((a + J_0) \pm (b + J_0)) = D(a + J_0) \pm D(b + J_0) = \overline{D}(a) \pm \overline{D}(b)$$

and

$$\begin{aligned}
\overline{D}(ab) &= D(ab + J_0)(b + J_0)) \\
&= (D(a + J_0))(b + J_0) + (a + J_0)(D(b + J_0)) \\
&= \overline{D}(a)(b + J_0) + (a + J_0)\overline{D}(b) \\
&= \overline{D}(a)b + a\overline{D}(b)
\end{aligned}$$

This is obvious that $\mathcal{A}$ is an $\mathfrak{A}$-module and the same as for $\frac{\mathcal{A}}{J_0}$. let X as a $\frac{\mathcal{A}}{J_0} - \mathfrak{A}-$ module with compatible actions in[**1**]. Hence

$$\overline{D}(\alpha.a) = D(\alpha.a + J_0) = D(\alpha.(a + J_0)) = \alpha * D(a + J_0) = \alpha * \overline{D}(a) = \alpha.\overline{D}(a)$$

and similarly $\overline{D}(a.\alpha) = \overline{D}(a).\alpha$.
In this case $\overline{D}$ is a module derivation and since $\mathcal{A}$ is module approximate amenable then there is $(x_\alpha) \in X^*$ such that

$$D(a + J_0) = \overline{D}(a) = lim_\alpha[a.(x_\alpha) - (x_\alpha).a] = lim_\alpha[(a + J_0).(x_\alpha) - (x_\alpha).(a + J_0)]$$

and hence D is inner □

THEOREM 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and Banach $\mathfrak{A}$-modules. If $\mathcal{A}$ be a module approximate amenable and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuouse epimorphism such that $\overline{\varphi(\mathcal{A})} = \mathcal{B}$, then $\mathcal{B}$ is module approximate amenable.*

PROOF. Let J_X be an $\mathcal{B}$-$\mathfrak{A}$-module. Since φ is epimorphism, then J_X is an $\mathcal{A}$-$\mathfrak{A}$-module . If $D : \mathcal{B} \longrightarrow J_X^\perp$ be a module derivation, then $Do\varphi : \mathcal{A} \longrightarrow J_X^\perp$ is module derivation. But $\mathcal{A}$ is module approximate amenable, then there exist net $\{x_\alpha\} \subseteq J_X^\perp$ such that for each $a \in \mathcal{A}$,

$$Do\varphi(a) = D(\varphi(a)) = lim_\alpha(\varphi(a).x_\alpha - x_\alpha.\varphi(a)).$$

Then $Do\varphi$ is inner. But $\overline{\varphi(\mathcal{A})} = \mathcal{B}$ and D is continouse, therefore D is inner. □

LEMMA 2.7. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $J_{\mathfrak{A}}$ is an closed ideal of $\mathcal{A}$. Module approximate amenability of $\mathcal{A}$ implies that of $\frac{\mathcal{A}}{J_{\mathfrak{A}}}$.*

PROOF. If $\pi : \mathcal{A} \longrightarrow \frac{\mathcal{A}}{J_{\mathfrak{A}}}$ defined by $\pi(a) = a + J_{\mathfrak{A}}$ for each $a \in \mathcal{A}$, then π is continuouse epimorphism and $\overline{\varphi(\mathcal{A})} = \mathcal{B}$. The result hold by Theorem 2.6. □

3. Semigroup Algebra

Recall that a discrete semigroup S is called *an inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $s^*ss^* = s$ and $ss^*s = s^*$. An element $e \in S$ is called *idempotent* if $e = e^* = e^2$. The set of all idempotents of S is denote by E. E is a commutative sub-semigroup of S with the natural order on E, defined by

$$e \leq d \qquad \Longleftrightarrow \qquad ed = e \qquad (e, d \in E).$$

$l^1(S)$ as a Banach algebra is a Banach $l^1(E)$ -module with compatible actions in [**1**]. The authors in [**17**] discuss approximate amenability of $l^1(S)$ with following actions

$$\delta_e.\delta_s = \delta_s \ \ and \ \ \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \ \ (s \in S, e \in E).$$

We consider right and left actions of $\ell^1(E)$ on $\ell^1(S)$ by

$$\delta_e.\delta_s = \delta_{es} = \delta_e * \delta_s \ \ and \ \ \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \ \ (s \in S, e \in E).$$

These actions make $\ell^1(S)$ a Banach $\ell^1(E_S)$-module.
Let $J_{\ell^1(S)}$ be the closed submodule of $\ell^1(S)$ generated by

$$\Big\{\delta_{set}-\delta_s : s,t\in S, e\in E_S\Big\}.$$

We define an equivalence relation on S as follows

$$s\approx t \qquad \Longleftrightarrow \qquad \delta_s-\delta_t\in J_{\ell^1(S)} \qquad (s,t\in S).$$

In [**1**], it is shown that S is amenable if and only if $\frac{S}{\approx}$ is amenable. The authors in [**6**], prove that for an inverse semigroup S the quotient semigroup $\frac{S}{\approx}$ is a discrete group. Also $\frac{l^1(S)}{J_{\ell^1(S)}}\cong l^1(\frac{S}{\approx})$.
Since $l^1(S)$ is a $l^1(E)$-module, then $\frac{l^1(S)}{J_{\ell^1(S)}}(\cong l^1(\frac{S}{\approx}))$ is a Banach $l^1(E)$-module by

$$\delta_e.(\delta_s+J_{\ell^1(S)})=\delta_{es}+J_{\ell^1(S)} \quad , \quad (\delta_s+J_{\ell^1(S)}).\delta_e=\delta_{se}+J_{\ell^1(S)}.$$

Let $s\in S$ and $e=s^*s\in E$, then $s=ss^*s=es$. Therefore

$$\delta_{se}-\delta_s=\delta_{se}-\delta_{es}\in J_{\ell^1(S)}.$$

So $se\approx s$ and this means that $\delta_e.\delta_{[s]}=\delta_{[se]}=\delta_{[s]}$. Similarly $\delta_{[s]}.\delta_e=\delta_{[es]}=\delta_{[s]}$. In this case we conclude $l^1(\frac{S}{\approx})$ is a commutative Banach $l^1(\frac{S}{\approx})$-$l^1(E)$-module.

Theorem 3.1. *Let S be an inverse semigroup. $l^1(\frac{S}{\approx})$ is module approximate amenable with canonical actions if and only if $l^1(\frac{S}{\approx})$ be an approximate amenable.*

Proof. We consider left and right actions of $l^1(E)$ on $l^1(\frac{S}{\approx})$ by $\delta_{[s]}*\delta_e=\delta_{[se]}=\delta_{[s]}, \quad \delta_e*\delta_{[s]}=\delta_{[es]}=\delta_{[s]} \quad (e\in E, s\in S)$.

Hence $l^1(\frac{S}{\approx})$ is commutative Banach $l^1(E)$-module. In the other word, we can say the left and right actions of $l^1(E)$ on $l^1(\frac{S}{\approx})$ is trivial. Thus

$$l^1(\tfrac{S}{\approx})\widehat{\otimes}_{l^1(E)}{}^1(\tfrac{S}{\approx})\cong l^1(\tfrac{S}{\approx})\bigotimes l^1(\tfrac{S}{\approx}).$$

Therefore $l^1(\frac{S}{\approx})$ is module approximate amenable if and only if $l^1(\frac{S}{\approx})$ is approximate amenable. ☐

Theorem 3.2. *Let S be an inverse semigroup, then $l^1(S)$ is module approximate amenable with canonical actions if and only if $l^1(\frac{S}{\approx})$ be a module approximate amenable*

Proof. We know that $\frac{l^1(S)}{J_{\ell^1(S)}}\cong l^1(\frac{S}{\approx})$. We proof that $l^1(S)$ is module approximate amenable if and only if $\frac{l^1(S)}{J_{\ell^1(S)}}$ is module approximate amenable.
Let left and right actions of $l^1(E)$ on X with $*$ (as an $\mathfrak{A}$-module).
Suppose that $\frac{l^1(S)}{J_{\ell^1(S)}}$ be a module approximate amenable and ${J_X}^{\perp}$ be a commutative $l^1(E)$-$l^1(S)$-module. Let $D:l^1(S)\longrightarrow {J_X}^{\perp}$ be a module derivation, then

$$\delta_{se}.x=\delta_s*(\delta_e.x)=\delta_s*(x.\delta_e)=(\delta_s*x).\delta_e=\delta_e.(\delta_s*x)=\delta_{es}.x \quad (e\in E, s\in S, x\in X).$$

Therefore $J_{\ell^1(S)}.X = 0$ and similarly $X.J_{\ell^1(S)} = 0$. Then X is commotative $\frac{l^1(S)}{J_{\ell^1(S)}}$-$l^1(E)$-module with the following module actions

$$(\delta_s + J_{\ell^1(S)}).x = \delta_s.x \quad x.(\delta_s + J_{\ell^1(S)}) = x.\delta_s,$$

for each $s \in S$ and $x \in X$. Since D is a module derivation, then

$$D(\delta_{es}) = \delta_e * D(\delta_s) \qquad D(\delta_{se}) = D(\delta_s) * \delta_e \quad (e \in E,\ s \in S).$$

But X is commutative $l^1(E)$-$l^1(S)$ -module. Hence $D(\delta_{es}) - D(\delta_{se}) = 0$. On the other hand

$$D(\delta_t).(\delta_{es} - \delta_{se}) = 0 = (\delta_{es} - \delta_{se}).D(\delta_h) \quad (t, h, s \in S,\ e \in E).$$

So

$$D(\delta_t(\delta_{es} - \delta_{se})\delta_h) = D(\delta_t).(\delta_{es} - \delta_{se})\delta_h + \delta_t(\delta_{es} - \delta_{se}).D(\delta_h) = 0,$$

therefore $D|_{J_{\ell^1(S)}} = 0$. Thus D induces a module derivation $\overline{D} : \frac{l^1(S)}{J_{\ell^1(S)}} \longrightarrow {J_X}^{\perp}$ that is defined by $\overline{D}(\delta_s + J_{\ell^1(S)}) = D(\delta_s)$.
If $\frac{l^1(S)}{J_{\ell^1(S)}}$ (as $l^1(E)$-module) be a module approximate amenable with canonical actions, then there exist net $(x_\alpha) \subseteq {J_X}^{\perp}$ such that

$$D(\delta_s) = \overline{D}(\delta_s + J_{\ell^1(S)}) = lim_\alpha[(\delta_s + J_{\ell^1(S)}).x_\alpha - x_\alpha(\delta_s + J_{\ell^1(S)})] = lim_\alpha(\delta_s.x_\alpha - x_\alpha.\delta_s).$$

Thus D is approximate inner and so $l^1(S)$ is module approximate amenable.
Conversely, let X be a commutative $\frac{l^1(S)}{J_{\ell^1(S)}}$-$l^1(E)$-module and $D : \frac{l^1(S)}{J_{\ell^1(S)}} \longrightarrow {J_X}^{\perp}$ be a module derivation, then X is a $l^1(S)$-$l^1(E)$-module with following module actions

$$\delta_s.x = (\delta_s + J_{\ell^1(S)}).x, \quad x.\delta_s = x.(\delta_s + J_{\ell^1(S)}) \quad (x \in X, s \in S).$$

Let $\overline{D} : l^1(S) \longrightarrow {J_X}^{\perp}$ be defined by $\overline{D}(\delta_s) = D(\delta_s + J_{\ell^1(S)})$ foe each $s \in S$, then $\overline{D}$ is module derivation. On the other hand $l^1(S)$ is module approximate amenable, then there exist a net $(x)_\alpha \subseteq {J_X}^{\perp}$ such that

$$\overline{D}(\delta_s) = lim_\alpha x_\alpha.\delta_s - \delta_s.x_\alpha = lim_\alpha(x_\alpha(\delta_s + J_{\ell^1(S)}) - (\delta_s + J_{\ell^1(S)}).x_\alpha) = \\ D(\delta_s + J_{\ell^1(S)}). \quad (s \in S).$$

Therefore D is approximate inner and thus $\frac{l^1(S)}{J_{\ell^1(S)}}$ is module approximate amenable. □

THEOREM 3.3. *Let S be an inverse semigroup with the set of idempotents E. $l^1(S)$ is $l^1(E)$-module approximate amenable with canonical actions if and onle if S is amenable.*

PROOF. By theorem 3.2, $l^1(S)$ is module approximate amenable (as a $l^1(E)$-module) with canonical actions if and only if $l^1(\frac{S}{\approx})$ is module approximate amenable as a $l^1(E)-$ module. Therefore $l^1(\frac{S}{\approx})$ is approximate amenability by theorem 3.1. Since $\frac{S}{\approx}$ is discrete group, then by [**18**, theorem 3.2], $l^1(\frac{S}{\approx})$ is amenable and so S is amenable. □

4. 2-weak module amenability of $l^1(S,\omega)$

Definition 4.1. Let S be a invers semigroup and $\Omega_{l^1(S,\omega)}$ be all of character on $l^1(S.\omega)$ w, then $\Omega_{l^1(S,\omega)}$ is called *0-property*

if $f=0$ when $\varphi(f)=0$, for each $\varphi\in\Omega_{l^1(S,\omega)}$ and $f\in l^1(S,\omega)$.

We know that a commutative Banach algebra is semisimple if and only if its Gelfand representation has trivial kernel. If $\Gamma: l^1(S,\omega)\longrightarrow C_0(\Omega_{l^1(S,\omega)})$ defined by $(f)=\widehat{f}$ be the Gelfand representation of $l^1(S,\omega)$, then

$$Ker\Gamma=\{f\in l^1(S,\omega):\widehat{f}(\varphi)=\varphi(f)=0\ for\ each\ \varphi\in\Omega_{l^1(S,\omega)}\}.$$

If $\Omega_{l^1(S,\omega)}$ has 0-*property* then $ker\Gamma=0$. In this case $l^1(S,\omega)$ is semisimple.

Theorem 4.2. *(Singer-Wermer theorem) Let $\mathfrak{A}$ be a commutative Banach algebra and D be a bounded derivation. Then D maps $\mathfrak{A}$ in to it's radical. In particualr if $\mathfrak{A}$ is semisimple then $D=0$.*

Proof. [**15**, theorem 2.7.20] □

Theorem 4.3. *Let S be a commutative inverse semigroup and ω be a weight on S such that $\omega(s)=\omega(s^*)$ for each $s\in S$. If the following three conditions are met*

(i) *For finite number $s\in S$ and $n\in N$, $\omega(s^*+(1-n)t)\geq 2\omega(s)$;*

(ii) *$\Omega_{l^1(S,\omega)}$ has $0-property$;*

(iii) *$inf_{n\in N}\frac{\omega(nt)}{n}=0$,*

For $t\in S$, then $l^1(S,\omega)$ as $l^1(E,\omega)$-module is 2-weak module amenable.

Proof. First we know that $l^1(S,\omega)^*$ is $l^1(S,\omega)-l^1(E,\omega)-$ module and the same as for $l^1(S,\omega)^{**}$. Let $D: l^1(S,\omega)\longrightarrow l^1(S,\omega)^{**}=l^\infty(S,\frac{1}{\omega})^*$ be a module derivation and $\parallel D\parallel\leq 1$.

If $\pi_\omega: l^1(S,\omega)^{**}\longrightarrow l^1(S,\omega)$ is a canonical projection then $\pi_\omega oD: l^1(S,\omega)\longrightarrow l^1(S,\omega)$ is a continuous derivation. Since $\Omega_{l^1(S,\omega)}$ has the $0-property$ then $l^1(S,\omega)$ is semisimple Banach algebra and by (4.2) , $\pi_\omega oD=0$ then $D(l^1(S,\omega))\subseteq Ker\pi_\omega$. To prove $D=0$ it suffices to shows that for each $\lambda\in l^1(S,\omega)^*$ we have $\langle D(\delta_t),\lambda\rangle=0$. We define $H_n=\{s\in S:\omega(s^*+(1-n)t)\geq 2\omega(s)\}$ and with (i) , H_n is finite. Define

$$\theta_n(s)=\begin{cases} 0 & \text{if } s\in H_n\\ (\lambda.\delta_{(1-n)t})(s) & \text{if } s\in S-H_n,\lambda\in l^1(E,\omega)\end{cases}$$

We have

$$sup_{s\in S}\frac{|\theta_n(s)|}{\omega(s)}=sup_{s\in S-H_n}\{\frac{\lambda(s^*+(1-n)t)}{\omega(s^*+(1-n)t)}.\frac{\omega(s^*+(1-n)t)}{\omega(s)}\leq 2\parallel\lambda\parallel$$

In this case $\parallel\theta_n\parallel\leq 2\parallel\lambda\parallel$ and so $\theta_n\in l^\infty(S,\frac{1}{\omega})$. In the other way

$$\langle \delta_{(1-n)t}.D(\delta_{nt}), \lambda\rangle = \langle D(\delta_{nt}), \lambda.\delta_{(1-n)t}\rangle = \langle D(\delta_{nt}), \theta_n\rangle.$$

Since for each $n \in Z$ we have $D(a) = \frac{1}{n}a^{1-n}D(a^n)$, therefore

$$\begin{aligned}\langle D(\delta_t), \lambda\rangle &= \langle \tfrac{1}{n}\delta_{(1-n)t}.D(\delta_{nt}), \lambda\rangle \\ &= \tfrac{1}{n}\langle \delta_{(1-n)t}.D(\delta_{nt}), \lambda\rangle \\ &= \tfrac{1}{n}\langle D(\delta_{nt}), \lambda.\delta_{(1-n)t}\rangle \\ &= \tfrac{1}{n}\langle D(\delta_{nt}), \theta_n\rangle.\end{aligned}$$

In this case

$$\mid \langle D(\delta_t), \lambda\rangle \mid = \frac{1}{n} \mid \langle D(\delta_{nt}), \theta_n\rangle \mid = \frac{1}{n} \parallel D(\delta_{nt}) \parallel\parallel \theta_n \parallel \leq \frac{1}{n} \parallel \delta_{nt} \parallel_\omega \parallel \theta_n \parallel \leq 2\frac{\omega(nt)}{n} \parallel \lambda \parallel .$$

But $inf_{n\in N}\frac{\omega(nt)}{n} = 0$, then $\langle D(\delta_t), \lambda\rangle = 0$. □

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