Applications of Integrals under A New Alternative of Differential and Comprehensive Calculation

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Abstract: *In the study of differential and integral calculus, integration requires knowledge of techniques to obtain a large number of integrals. The central idea consists in this technique, to recognize the type of fundamental integral. This technique is common in several textbooks at the undergraduate level and requires the student to have a good theoretical and mathematical knowledge. Thinking along this path, we present in this article the development of a theorem that can be used in calculating primitives without resorting to techniques known in textbooks. In this case, the study is restricted in integrals that have quadratic function in the integrand of the denominator with exponent ½ and in the numerator, related function of the type px + q. For this type of parameters, we obtain primitives that depend on the nature of the roots of the quadratic function and once obtained can literally be applied in several cases of integrals of the same nature, without, however, resort to techniques for changes of variables . To better approach this alternative method, we discuss the root cases of the quadratic function and with the results of each primitive, apply in examples for a better understanding of the method.*
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I. Introduction To The Calculation Study

The contributions of mathematicians to the birth of Calculus were decisive and used concepts of Calculus to solve various problems and in different areas of knowledge, especially with applications in the area of economics [1-3]. The Calculus can be divided into two parts: one related to derivatives or Differential Calculus and another part related to integrals. The so-called top-level computation has been a powerful tool to aid in a large number of problems requiring this knowledge [4-7],

The historical trajectory of calculus can be attributed to Newton, however, the contributions of mathematicians to the birth of Calculus are innumerable. Many of them, even if imprecise or not rigorous, already used concepts of Calculus to solve various problems involving physics in the study of gravitation, historically [8-10].

With the advent of new advances, computation has been useful in many other scientific segments and has contributed to explain many problems related to physical and chemical properties with the use of computational resources. In this sense the calculation has been applied and studied with the use of methods and developments that lead to results of functions for the calculation of certain physical quantities and with the insertion of other methodologies easy to learn [11-14].

1.1 General Theory Of Fundamental Integral

Every type integral,

$$I = \int \frac{(px+q)dx}{(ax^2+bx+c)^m}. \quad (1)$$

It can be considered as the sum of two integrals $I = I_1 + I_2$.
at where,

$$I_1 = \frac{p}{2a(1-m)} e^{y\left(\frac{1-m}{m}\right)}. \quad (2)$$

and

$$I_2 = K_1 \int \frac{1}{\sqrt{1+x_1^2}} x_1^{1-2m}\, dx_1. \quad (3)$$

With

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^{\frac{y}{2m}}. \quad (4)$$

And

$$K_1 = \frac{\sqrt{\Delta}}{2a}\left(\frac{2\sqrt{a}}{\sqrt{\Delta}}\right)^{2m} \cdot \left(\pm q \mp \frac{Pb}{2a}\right). \quad (5)$$

1.2 Demonstration Of Theorem

Be the integral,

$$I = \int \frac{(px+q)dx}{(ax^2+bx+c)^m}. \quad (1)$$

To get a primitive function of the integrand parameters, let's use the following artífice

$$.(ax^2+bx+c)^m = e^y \rightarrow ax^2+bx+c = e^{\frac{y}{m}}. \quad (2)$$

At this point, a question may arise: Why express an equality between a quadratic function and an exponential?

We can consider that the main objective is to find the primitive given literally, for any type of integral given by equation (1) and being the exponential a function that has the same integral or derived less than a constant, it is possible to replace the integrand of denominator given in (2), by the exponential considered in (2), considering an equality between the quadratic and exponential functions. Thus deriving (2), we obtain,

$$(2ax+b)dx = \frac{dy}{m} e^{\frac{y}{m}} \rightarrow dx = \frac{e^{\frac{y}{m}}}{m} \frac{1}{2ax+b} dy. \quad (3)$$

Taking (3) into (1), it follows that,

$$I = \int \frac{(Px+q)}{(2ax+b)} \frac{1}{m} \frac{e^{\frac{y}{m}}}{e^y} dy \rightarrow I = \int \frac{(Px+q)}{(2ax+b)} \frac{1}{m} e^{y\left(\frac{1}{m}-1\right)} dy. \quad (4)$$

Taking again equation (2) and bringing the terms from the 2nd member to the 1st member, we obtain,

$$ax^2+bx+c = e^{\frac{y}{m}} \rightarrow ax^2+bx+\left(c-e^{\frac{y}{m}}\right) = 0. \quad (5)$$

We must consider that the equation in (5) must be regarded as a quadratic function, although it has the presence of the exponential e ^ (y / m). Thus, bycalculating Δ_1 in (5), weobtain, afterdevelopment,

$$\Delta_1 = \Delta\left[1+\left(\frac{2\sqrt{a}}{\sqrt{\Delta}} e^{\frac{y}{2m}}\right)^2\right].$$

Observe thesteps,

$$\Delta_1 = b^2-4ac = b^2-4a.\left(c-e^{\frac{y}{m}}\right) = b^2-4ac+4ae^{\frac{y}{m}}$$

$$\Delta_1 = \Delta + 4ace^{\frac{y}{m}} = \Delta\left(1+\frac{4a}{\Delta} e^{\frac{y}{m}}\right)$$

$$\Delta_1 = \Delta\left[1+\left(\frac{2\sqrt{a}}{\sqrt{\Delta}} e^{\frac{y}{2m}}\right)^2\right]. \quad (6)$$

Whereas

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^{\frac{y}{2m}}. \quad (7)$$

And extracting the root of Δ_1 , comes that,

$$\sqrt{\Delta_1} = \pm\sqrt{\Delta}\sqrt{(1+x_1^2)}. \quad (8)$$

As

$$x = \frac{-b \pm \sqrt{\Delta_1}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}\sqrt{1+x_1^2}}{2a} = V_x \pm \frac{\sqrt{\Delta}\sqrt{1+x_1^2}}{2a}.$$

at where $V_x = -\frac{b}{2a}$is represented as the abscissa of the vertex of the quadratic function

Bringing the first term from the 2nd member to the 1st member and after a development, we obtain the result,

$$2ax + b = \pm\sqrt{\Delta}\sqrt{1+x_1^2} \rightarrow V_x \pm \frac{\sqrt{\Delta}\sqrt{1+x_1^2}}{2a}. \quad (9)$$

Taking equation (9) and (4) in the equation given by (1), we obtain after the developments that,

$$I_1 = K_1.\int \frac{1}{\sqrt{1+x_1^2}}.x_1{}^{1-2m}dx_1.$$

And

$$I_2 = \frac{P}{2a(1-m)} e^{y\left(\frac{1-m}{m}\right)}.$$

atwhere

$$K_1 = \frac{\sqrt{\Delta}}{2a}\left(\frac{2\sqrt{a}}{\sqrt{\Delta}}\right)^{2m}\left(\pm q \mp \frac{Pb}{2a}\right).$$

And

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^{\frac{y}{2m}}.$$

Note the following passes, from (9) and (4) to (1), to obtain the previous results,

$$I = \int \frac{(px+q)}{\pm\sqrt{\Delta}\sqrt{1+x_1^2}}.\frac{1}{m} e^{y\left(\frac{1-m}{m}\right)} dy. \quad (10)$$

Distributing the terms, we obtain,

$$I = \int \left[p.\left(V_x \pm \frac{\sqrt{\Delta}\sqrt{1+x_1^2}}{2a}\right) + q\right].\frac{1}{\pm m\sqrt{\Delta}\sqrt{(1+x_1^2)}} e^{y\left(\frac{1-m}{m}\right)} dy \quad (11)$$

Distributing the terms in (11). This is,

$$I = \int \frac{(\pm pV_x)}{m\sqrt{\Delta}\sqrt{1+x_1^2}} e^{y\left(\frac{1-m}{m}\right)} dy + \int \frac{p}{2am} e^{y\left(\frac{1-m}{m}\right)} dy \pm \frac{q}{\sqrt{\Delta}m}\int \frac{e^{y\left(\frac{1-m}{m}\right)}}{\sqrt{1+x_1^2}} dy.$$

By grouping the constant terms and extracting from the integral signal, we have,

$$I = \left(\frac{\pm q}{\sqrt{\Delta}m} \pm \frac{pV_x}{m\sqrt{\Delta}}\right)\int \frac{1}{\sqrt{1+x_1^2}} e^{y\left(\frac{1-m}{m}\right)} dy + \int \frac{p}{2am} e^{y\left(\frac{1-m}{m}\right)} dy.$$

Or yet,

$$I = \frac{1}{m\sqrt{\Delta}}\left(\pm q \mp \frac{pb}{2a}\right)\int \frac{1}{\sqrt{1+x_1^2}} e^{y\left(\frac{1-m}{m}\right)} dy + \int \frac{p}{2am} e^{y\left(\frac{1-m}{m}\right)} dy. \quad (12)$$

Denoting K for the first term of equation (12) and taking, $V_x = -\frac{b}{2a}$, we get that,

$$K = \frac{1}{m\sqrt{\Delta}}\left(\pm q \mp \frac{Pb}{2a}\right). \quad (13)$$

Taking (13) into (12), comes,

$$I = K.\int \frac{1}{\sqrt{1+x_1^2}} e^{y\left(\frac{1-m}{m}\right)} dy + \int \frac{p}{2am} e^{y\left(\frac{1-m}{m}\right)} dy. \quad (14)$$

Denoting in equation (14), I_2. That is, being

$$I_2 = \frac{P}{2am}\int e^{y\left(\frac{1-m}{m}\right)} dy. \quad (15)$$

Making a change of variable in the exponential of the equation in (1), it follows that,

$$y\left(\frac{1-m}{m}\right) = u \rightarrow dy = \frac{du.m}{(1-m)}. \quad (16)$$

Taking (16) in (15), we have that,

$$I_2 = \frac{p}{2am}.\frac{m}{(1-m)}\int e^u.du = \frac{P}{2a(1-m)}e^u.$$

Thus, the integral in (15) has the following primitive, since the exponential remains less than signal, as expected. Like this,

$$I_2 = \frac{p}{2a(1-m)} e^{y\left(\frac{1-m}{m}\right)}. \quad (17)$$

Where the exponential of the relation in (17) with the relation given in (2) must be related to find the primitive as a function of the variable x. Let now be the first term of the expression given in (14), that is,

$$I_1 = K.\int \frac{1}{\sqrt{1+x_1^2}} e^{y\left(\frac{1-m}{m}\right)} dy. \quad (18)$$

Let the expression in (7), that is,

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^{\frac{y}{2m}}.$$

Differentiating the expression (7) in the variable x_1 as a function of y, we obtain that

$$dx_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}}.\frac{dy}{2m} e^{\frac{y}{2m}} \rightarrow dy = \frac{\sqrt{\Delta}md\,x_1}{\sqrt{a}} e^{-\frac{y}{2m}}. \quad (19)$$

Taking (19) in (18), comes,

$$I_1 = K\int \frac{1}{\sqrt{1+x_1^2}}.e^{y\left(\frac{1-m}{m}\right)}.\frac{\sqrt{\Delta}m}{\sqrt{a}} e^{-\frac{y}{2m}} dx_1 \rightarrow I_1 = K.\frac{\sqrt{\Delta}m}{\sqrt{a}}\int \frac{1}{\sqrt{1+x_1^2}} e^{y\left(\frac{1-m}{m}-\frac{1}{2m}\right)} dx_1.$$

As $\frac{\sqrt{\Delta}m}{\sqrt{a}}$is a constant term, was taken from under the sign of integral, thus,

$$I_1 = \frac{K\sqrt{\Delta}m}{\sqrt{a}} \int \frac{1}{\sqrt{1+x_1^2}} e^{y\left(\frac{2-2m-1}{2m}\right)} dx_1 \rightarrow$$

$$I_1 = \frac{K\sqrt{\Delta}m}{\sqrt{a}} \int \frac{1}{\sqrt{1+x_1^2}} e^{y\left(\frac{1-2m}{2m}\right)} dx_{1.} \tag{20}$$

Taking again the relation in (7), we obtain that,

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^{\frac{y}{2m}} \rightarrow e^{\frac{y}{2m}} = \frac{x_1\sqrt{\Delta}}{2\sqrt{a}}. \tag{21}$$

By placing the natural logarithmic on both limbs,

$$\frac{y}{2m} = \ln\left(\frac{x_1\sqrt{\Delta}}{2\sqrt{a}}\right) \rightarrow y = \ln\left(\frac{x_1\sqrt{\Delta}}{2\sqrt{a}}\right)^{2m}. \tag{22}$$

Substituting y given by relation (22) into expression (20), it follows that,

$$I_1 = \frac{K\sqrt{\Delta}m}{\sqrt{a}} \int \frac{1}{\sqrt{1+x_1^2}} e^{\left(\frac{1-2m}{2m}\right).\ln\left(\frac{x_1\sqrt{\Delta}}{2\sqrt{a}}\right)^{2m}} dx_1. \tag{23}$$

Considering the exponential given in (23) and using the property of the logarithmic,

$$e^{\left(\frac{1-2m}{2m}\right).\ln\left(\frac{x_1\sqrt{\Delta}}{2\sqrt{a}}\right)^{2m}} = e^{\ln\left[\left(\frac{x_1\sqrt{\Delta}}{2\sqrt{a}}\right)^{2m}\right]^{\frac{1-2m}{2m}}} = \left[\left(\frac{x_1\sqrt{\Delta}}{2\sqrt{a}}\right)^{2m}\right]^{\frac{1-2m}{2m}}. \tag{24}$$

Taking (24) into (23), it follows that,

$$I_1 = \frac{K\sqrt{\Delta}m}{\sqrt{a}} \int \frac{1}{\sqrt{1+x_1^2}} \left(\frac{x_1\sqrt{\Delta}}{2\sqrt{a}}\right)^{1-2m} dx_{1.} \tag{25}$$

Taking out the integral signal from the terms in (25), it follows that,

$$I_1 = \frac{K\sqrt{\Delta}m}{\sqrt{a}} . \left(\frac{\sqrt{\Delta}}{2\sqrt{a}}\right)^{1-2m} \int \frac{1}{\sqrt{1+x_1^2}} {x_1}^{1-2m} dx_{1.} \tag{26}$$

Denoting by K_1o constant term outside the signal of the integral in relation (26), we obtain that

$$K_1 = \frac{K\sqrt{\Delta}m}{\sqrt{a}} \left(\frac{\sqrt{\Delta}}{2\sqrt{a}}\right)^{1-2m}. \tag{27}$$

And the integral in (26) with the substitution of the relation (27), becomes,

$$I_1 = K_1 . \int \frac{1}{\sqrt{1+x_1^2}} . {x_1}^{1-2m} dx_1. \tag{28}$$

Which represents the second solution of the integral given by (1)?

Thus, considering the integral (1), and the results given by (15) and (28), we obtain that,

$$I = \int \frac{(px+q)dx}{(ax^2+bx+c)^m} = I_1 + I_2 = K_1 . \int \frac{1}{\sqrt{1+x_1^2}} . {x_1}^{1-2m} dx_1 + \frac{P}{2a(1-m)} e^{y\left(\frac{1-m}{m}\right)}. \tag{29}$$

What completes the theorem. The integral given by (29) makes it possible to solve a large number of integral and can be used to obtain primitives, aiding in precise and less complicated solutions when compared to the usual solution method. The theorem given by (29) can be used to obtain important primitives for each value assigned to constant m. In this article we will use the relation (29) to apply in some mathematical problems in order to show how much this alternative method is relevant. Considering also the expression given by (27) and the relation given by (13), we obtain K_1 as a function of the parameters,

$$K_1 = \frac{\sqrt{\Delta}m}{\sqrt{a}}.\left(\frac{\sqrt{\Delta}}{2\sqrt{a}}\right)^{1-2m}.\frac{1}{m\sqrt{\Delta}}\left(\pm q \mp \frac{Pb}{2a}\right) \rightarrow K_1 = \frac{1}{\sqrt{a}}\left(\frac{\sqrt{\Delta}}{2\sqrt{a}}\right)^{1-2m}.\left(\pm q \mp \frac{Pb}{2a}\right) \rightarrow$$

$$K_1 = \frac{1}{\sqrt{a}}\left(\frac{\sqrt{\Delta}}{2\sqrt{a}}\right)\left(\frac{2\sqrt{a}}{\sqrt{\Delta}}\right)^{2m}\left(\pm q \mp \frac{Pb}{2a}\right) \rightarrow$$

$$K_1 = \frac{\sqrt{\Delta}}{2a}\left(\frac{2\sqrt{a}}{\sqrt{\Delta}}\right)^{2m} K. \tag{30}$$

At where,

$$K = \left(\pm q \mp \frac{Pb}{2a}\right)$$

The relationship given by (30) represents a coefficient that can be real (for positive Δ) and complex (for negative Δ). In this way, it can be considered real or complex and depends on the type of integral to be calculated. In the following section, we will study two cases for the value assigned to K. In both cases we will use m = 1 taking into account $\Delta > 0$ and $\Delta < 0$, which will be addressed in the following topic.

1.3 Consequences of the Fundamental Theorem for $m = 1/2$

In this topic we will use the results obtained from the fundamental theorem, the use of primitives associated with the parameter m = 1/2. However, it is necessary to do the demonstration for each primitive considering the root discussion of the quadratic function that appears in the integrand of the denominator of the fundamental integral and in the case of m = 1/2. First, let's get the general integral and from this integral, make the particularities.

The integral of the expression by (1), has as its solution the primitive when $m = 1/2$:

$$I = \frac{P}{a}\left(\frac{1}{\sqrt{ax^2+bx+c}}\right)^{-1} + K_1 \ln\left(\sqrt{1+x_1^2} + x_1\right). \tag{31}$$

With

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^y. \tag{32}$$

and

$$K_1 = \frac{1}{\sqrt{a}}\left(\pm q \mp \frac{Pb}{2a}\right). \tag{33}$$

Demonstration

Taking the value of m = 1/2 in the integral given by (1), we obtain

$$I = \int \frac{(px+q)dx}{\sqrt{ax^2+bx+c}}. \tag{34}$$

Therefore, for the relation given in (30), we obtain,

$$K_1 = \frac{\sqrt{\Delta}}{2a}\left(\frac{2\sqrt{a}}{\sqrt{\Delta}}\right)^1\left(\pm q \mp \frac{pb}{2a}\right) \rightarrow$$

$$K_1 = \frac{\sqrt{a}}{a}\left(\pm q \mp \frac{pb}{2a}\right) = \pm\frac{q}{\sqrt{a}} \mp \frac{pb}{2a\sqrt{a}} \text{ ou}$$

$$K_1 = \frac{1}{\sqrt{a}}\left(\pm q \mp \frac{Pb}{2a}\right). \quad (35)$$

According to the relation given by (17), it follows that,

$$I_2 = \frac{P}{2a(1-m)} e^{y\left(\frac{1-m}{m}\right)}.$$

For$m = 1/2$, get as a result,

$$I_2 = \frac{P}{a} e^y. \quad (36)$$

In agreement with a relationship (2), we obtain for$m = 1/2$,

$$(ax^2 + bx + c)^{1/2} = e^y. \quad (37),$$

Therefore, taking (37) and, (36), see what,

$$I_2 = \frac{p}{a}(ax^2 + bx + c)^{1/2}. \quad (38)$$

Taking a relaçãoem (28) e considering m = 1/2, it implies

$$I_1 = K_1. \int \frac{1}{\sqrt{1+x_1^2}}. dx_1. \quad (39)$$

Let's make the following change of variable,

$$\operatorname{tg}\theta = x_{1.} \quad (40)$$

Deriving the relation in (40), we obtain

$$dx_1 = \sec^2\theta d\theta \quad (41)$$

Taking the relation (40) and (41) into (39), we obtain the following integral after realizing the developments,

$$I_1 = K_1. \int sec\theta\, d\theta \quad (42)$$

The solution of the integral of the expression given by (42), according to the theory of integral calculus obeys the relation,

$$I_1 = K_1. \ln|\sec\theta + \operatorname{tg}\theta| \quad (43)$$

From the relation (40), we extract for the values of sinθ and cosθ the expressions,

$$\operatorname{sen}\theta = \frac{x_1}{\sqrt{1+x_1^2}} \quad (44)$$

And

$$\cos\theta = \frac{1}{\sqrt{1+x_1^2}} = \frac{1}{sec\theta} \quad (45)$$

Taking (44) and (45) into (43), we obtain that,

$$I = K_1. \ln\left(\sqrt{1+x_1^2} + x_1\right) \quad (46)$$

With

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^y \quad (47)$$

Therefore, the solution will be,

$$I_1 + I_2 = = K_1 . \ln\left(\sqrt{1+x_1^2} + x_1\right) + \frac{p}{a}(ax^2+bx+c)^{1/2}. \qquad (48)$$

We must now discuss the cases based on the first term of the integral.

As

$$K_1 = \frac{1}{\sqrt{a}}\left(\pm q \mp \frac{Pb}{2a}\right). \qquad (49)$$

considered K_1 the first sign, we must,

$$K_1 = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right). \qquad (50)$$

Let us operate the integral given by,

$$I_1 = K_1 . \ln\left(\sqrt{1+x_1^2} + x_1\right) \qquad (51)$$

And

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}} e^y \qquad (52)$$

With

$$(ax^2+bx+c)^{1/2} = e^y \qquad (53)$$

Where substituting (53) into (52), it is obtained that,

$$x_1 = \frac{2\sqrt{a}}{\sqrt{\Delta}}(ax^2+bx+c)^{1/2} \qquad (54)$$

Raising (54) the square and summing with the unit and extracting the root, we obtain that,

$$x_1^2 = \frac{4.a(ax^2+bx+c)}{\Delta}$$

And

$$x_1^2 + 1 = \frac{4.a^2x^2 + 4abx + 4ac + \Delta}{\Delta} = \frac{4.a^2x^2 + 4abx + 4ac + b^2 - 4ac}{\Delta}$$

So,

$$x_1^2 + 1 = \frac{4.a^2x^2 + 4abx + 4ac + \Delta}{\Delta} = \frac{4.a^2x^2 + 4abx + b^2}{\Delta} = \frac{(2ax+b)^2}{\Delta}$$

Extracting the root, one has that,

$$\sqrt{x_1^2+1} = \frac{|2ax+b|}{\sqrt{\Delta}} \qquad (55)$$

Taking (55), (54) and (50) into (51), we obtain that,

$$I_1 = \left[\frac{1}{\sqrt{a}}\left(\pm q \mp \frac{Pb}{2a}\right)\right] . \ln\left(\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2\sqrt{a}}{\sqrt{\Delta}}(ax^2+bx+c)^{1/2}\right) \qquad (56)$$

Therefore, the integral will have the following solution according to expressions (38) and (56)

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right) . \ln\left(\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2\sqrt{a}}{\sqrt{\Delta}}(ax^2+bx+c)^{1/2}\right) + \frac{p}{a}(ax^2+bx+c)^{1/2} \qquad (57)$$

Where it considers the first sign of k in the expression (56). The expression given by (57) shows that it is necessary to perform a discussion in the primitive, since in the first term of that integral, there is a square root for the coefficient a andΔ to be analyzed and discussed. This shows that for each discussion, there will be a new primitive to be considered. This will be seen in the following subtopics.

1.3.1 Private cases: When $a < 0$ e $\Delta > 0$

Then, taking the expression in (56), we obtain that, as$a < 0$, So$\sqrt{a} = i.\sqrt{-a}$

Thus (57), we obtain that,

$$I_1 = -i\frac{1}{\sqrt{-a}} . \left(q - \frac{Pb}{2a}\right) \ln\left(\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2i\sqrt{-a}}{\sqrt{\Delta}}(ax^2+bx+c)^{1/2}\right). \qquad (58)$$

Because

$$i\sqrt{-a} = \sqrt{a}$$

And

$$\frac{1}{i\sqrt{-a}} = -\frac{i}{\sqrt{-a}}$$

Since there is a complex Logarithm, we must relate to the trigonometric function. We consider that

$$\frac{|2ax+b|}{\sqrt{\Delta}} = sen\varphi = h \quad (59)$$

From Euler's relation, we obtain,

$$sen\varphi = h = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \quad (60)$$

Developing (60), we obtain that,

$$h = \frac{e^{2i\varphi} - 1}{2i.e^{i\varphi}} \quad (61)$$

Where we multiply in the numerator and denominator by e ^ iφ
Thus, we must,

$$2i.e^{i\varphi}.h = e^{2i\varphi} - 1 \quad (62)$$

Denoting

$$e^{i\varphi} = \omega, \quad (63)$$

We must, after clearing the terms,

$$\omega^2 - 2i.\omega.h - 1 = 0 \quad (64)$$

That represents an equation of the 2nd in ω with discriminant given by,

$$\Delta = -4h^2 + 4 = 4(1 - h^2) \quad (65),$$

Therefore, the roots will be,

$$\omega = \frac{ih \pm \sqrt{1-h^2}}{1} \quad (66)$$

Where we will take into account only the root with positive sign. So we have to

$$\omega = ih + \sqrt{1 - h^2} \quad (67)$$

In view of the expression given by (59), it follows that,

$$h = \frac{|2ax+b|}{\sqrt{\Delta}} \quad (68)$$

We have after development that,

$$1 - h^2 = (-4x^2a^2 - 4abx - b^2 + b^2 - 4ac)/\Delta$$

$$1 - h^2 = \frac{-4a^2x^2 - 4abx - 4ac}{\Delta} = \frac{-4a(ax^2 + bx + c)}{\Delta} =$$

Extracting the root, we obtain that,

$$\sqrt{1 - h^2} = \frac{2i\sqrt{a}}{\sqrt{\Delta}}(ax^2 + bx + c)^{1/2} \quad (69)$$

Taking (67), (68) in (65), we obtain that

$$\omega = i\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2i\sqrt{a}}{\sqrt{\Delta}}(ax^2 + bx + c)^{1/2} \quad (70)$$

Given that,

$$e^{i\varphi} = \omega \quad (71)$$

Therefore, taking (70) into (71), we have,

$$e^{i\varphi} = \left| i\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2i\sqrt{a}(ax^2+bx+c)^{1/2}}{\sqrt{\Delta}} \right| \quad (72)$$

Considering the logarithm in both members, we obtain,

$$i\varphi = Ln\left| i\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2i\sqrt{a}(ax^2+bx+c)^{1/2}}{\sqrt{\Delta}} \right| \quad (73)$$

As the

$$|i| = 1$$

Thus, we obtain that,

$$\varphi = -iLn\left| \frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2\sqrt{a}(ax^2+bx+c)^{1/2}}{\sqrt{\Delta}} \right| \quad (74)$$

As< 0 , We have to,

$$\varphi = -iLn\left|\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2i\sqrt{-a}(ax^2+bx+c)^{1/2}}{\sqrt{\Delta}}\right|$$

Based on the expression in (59). This is,

$$sen\varphi = h \quad (75)$$

Soon,

$$arsenh = \varphi \quad (76)$$

Considering the expression given by (67), it follows that,

$$\text{arsen}(\frac{|2ax+b|}{\sqrt{\Delta}}) = -iLn\left|\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2i\sqrt{-a}(ax^2+bx+c)^{1/2}}{\sqrt{\Delta}}\right| \quad (77)$$

Taking the equation given by (58). This is,

$$I_1 = -i\frac{1}{\sqrt{-a}}.\left(q - \frac{Pb}{2a}\right)\ln\left(\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2i\sqrt{-a}}{\sqrt{\Delta}}(ax^2+bx+c)^{1/2}\right) \quad (78)$$

Comparing with the expression given by (78), we obtain the important relation,

$$I_1 = \frac{1}{\sqrt{-a}}\left(q - \frac{Pb}{2a}\right)\text{arsen}(\frac{|2ax+b|}{\sqrt{\Delta}}) (79)$$

Thus, the solution of the restricted integral to the case that a <0 and Δ> 0, has the following primitive,

$$I = \frac{p}{a}(ax^2+bx+c)^{1/2} + \frac{1}{\sqrt{-a}}\left(q - \frac{Pb}{2a}\right)\text{arsen}(\frac{|2ax+b|}{\sqrt{\Delta}} (80)$$

Therefore, it is verified that the primitive given by (80) is general and can be applied whenever the condition of a <0 and Δ> 0 is satisfied. Let's look at the following examples and an application at arc length.

Examples of applications.

Example 1
Calculate the integral,

$$I = \int \frac{x+2}{\sqrt{4x-x^2}}dx$$

In this case, the parameters are given by,

$p = 1, q = 2, a = -1, b = 4 \ \ ec = 0e\Delta = 16$

According to the comparison of expression (34) with the integral given in the example.
Taking the expression (78). This is,

$$I = \frac{p}{a}(ax^2+bx+c)^{1/2} + \frac{1}{\sqrt{-a}}\left(q - \frac{pb}{2a}\right)\text{arsen}(\frac{|2ax+b|}{\sqrt{\Delta}}$$

Taking the values of the parameters, we obtain that,

$$I = -(4x-x^2)^{\frac{1}{2}} + 4\text{arsen}(\frac{|-x+2|}{2})$$

Or

$$I = -(4x-x^2)^{\frac{1}{2}} + 4\text{arsen}(\frac{x-2}{2})$$

Example2
Calculate the integral,

$$I = \int \frac{x+3}{\sqrt{5-4x-x^2}}dx$$

$$p = 1, q = 3, a = -1, b = -4, c = 5 \ e \ \Delta = 36$$

Taking the expression (79). This is,

$$I = \frac{p}{a}(ax^2 + bx + c)^{1/2} + \frac{1}{\sqrt{-a}}\left(q - \frac{Pb}{2a}\right)\text{arsen}(\frac{|2ax + b|}{\sqrt{\Delta}}$$

And taking the values, we obtain that,

$$I = -(5 - 4x - x^2)^{\frac{1}{2}} + \text{arsen}(\frac{x + 2}{3})$$

Example 3

Calculate the integral,

$$I = \int \frac{dx}{\sqrt{4 - x^2}}$$

$$p = 0, q = 1, a = -1, b = 0, c = 4 \ e \ \Delta = 16$$

Using the expression in (81), it follows that,

$$I = \frac{p}{a}(ax^2 + bx + c)^{1/2} + \frac{1}{\sqrt{-a}}\left(q - \frac{Pb}{2a}\right)\text{arsen}(\frac{|2ax + b|}{\sqrt{\Delta}}$$

The first integral cancels out, since p = 0. We have only the second integral. This is,

$$I = \frac{1}{\sqrt{-a}}\left(q - \frac{Pb}{2a}\right)\text{arsen}(\frac{|2ax + b|}{\sqrt{\Delta}}$$

Taking the values of the parameters,

$$I = \text{arsen}(\frac{x}{2})$$

Example 4

Length of circumference.

Seals the equation of a circle in Cartesian coordinates

$$x^2 + y^2 = r^2$$

Determine the length of this circumference

Solution

$$x^2 + y^2 = r^2 \rightarrow y = \sqrt{r^2 - x^2} \rightarrow \frac{dy}{dx} = -\frac{2x}{2\sqrt{r^2 - x^2}} = -\frac{x}{\sqrt{r^2 - x^2}}$$

Taking the last expression before the square, we obtain,

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2 - x^2},$$

Since the length of an arc is given by the expression (STEWART, 2006)

$$S = \int \sqrt{1 + [(f'(x))]^2} dx$$

As

$$f'(x) = -\frac{x}{\sqrt{r^2 - x^2}}$$

Taking in the expression of S, it follows that,

$$S = \int \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx$$

Ou ajeitando os termos sob o sinal do radical, vem que,

$$S = \int \frac{r}{\sqrt{r^2 - x^2}} dx$$

According to the expression (4), we have the following parameters,

$a = -1, b = 0, p = 0, q = r, \Delta = 4r^2$

In this case, we have, $a < 0\ e\ \Delta > 0$

So the solution is given by expression (80). Thisis,

$$I = \frac{p}{a}(ax^2 + bx + c)^{1/2} + \frac{1}{\sqrt{-a}}\left(q - \frac{Pb}{2a}\right)\text{arsen}(\frac{|2ax + b|}{\sqrt{\Delta}}$$

The first one cancels out (p = 0). We have for the second term,

$$I = \frac{1}{\sqrt{-a}}\left(q - \frac{Pb}{2a}\right)\text{arsen}(\frac{|2ax + b|}{\sqrt{\Delta}}$$

Taking the given values, we obtain that,

$$I = r.\text{arsen}(\frac{x}{r})$$

Taking into account the limit of integration,

$$I = r \text{ arc sen}\left(\frac{x}{r}\right) |_0^r$$

or

$$I = r.(\text{arc sen } 1 - \text{arc sen } 0)$$

$$S = r.\left(\frac{\pi}{2} - 0\right) \rightarrow S = \frac{\pi r}{2}$$

Therefore, the length of the circumference will be:

$$S_T = 2\pi r$$

1.3.2 When$a > 0$ e $\Delta < 0$

Using the expression given by (57), we obtain that,

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right).\ln\left|\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2\sqrt{a}\left(ax^2+bx+c\right)^{1/2}}{\sqrt{\Delta}}\right| + \frac{p}{a}(ax^2 + bx + c)^{1/2} \tag{81}$$

As Δ<0, the expression in (81) becomes,

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right).\ln\left|\frac{|2ax+b|}{i\sqrt{-\Delta}} + \frac{2\sqrt{a}\left(ax^2+bx+c\right)^{1/2}}{i\sqrt{-\Delta}}\right| + \frac{p}{a}(ax^2 + bx + c)^{1/2} \tag{82}$$

Therefore, being

$$|i| = 1 \tag{83}$$

We have to,

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right).\ln\left|\frac{|2ax+b|}{\sqrt{-\Delta}} + \frac{2\sqrt{a}(ax^2+bx+c)^{1/2}}{\sqrt{-\Delta}}\right| + \frac{p}{a}(ax^2+bx+c)^{1/2} \quad (84)$$

Example1

Calcule a Integral

$$I = \int \frac{x+2}{\sqrt{x^2+9}}dx$$

Seja

$$p = 1, q = 2, a = 1 \text{ e } c = 9 \text{ com } b = 0, \Delta = -36$$

Como

$$a = 1 \text{e} \Delta = -36$$

Levando os parâmetros na expressão (84), obtemos que,

$$I = \frac{1}{\sqrt{1}}\left(2 - \frac{0}{2.1}\right).\ln\left|\frac{|2x+0|}{\sqrt{36}} + \frac{2\sqrt{1}(x^2+9)^{1/2}}{\sqrt{36}}\right| + \frac{1}{1}(x^2+9^{1/2}$$

Assim, obtemos que,

$$I = \sqrt{x^2+9} + 2\ln\left(\frac{x+\sqrt{x^2+9}}{3}\right)$$

Example 2

Calculate the integral,

$$I = \int \frac{(-2x+1)}{\sqrt{x^2+1}}dx \rightarrow$$

We have the following parameters,

$$p = -2, q = 1, a = 1, b = 0, c = 1, \Delta = -4$$

Using the expression given by (84). This is,

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right).\ln\left|\frac{|2ax+b|}{\sqrt{-\Delta}} + \frac{2\sqrt{a}(ax^2+bx+c)^{1/2}}{\sqrt{-\Delta}}\right| + \frac{p}{a}(ax^2+bx+c)^{1/2}$$

Taking the values of the parameters, we obtain that,

$$I = 1(1-0).\ln\left|\frac{|2x-0|}{\sqrt{4}} + \frac{2\sqrt{1}(x^2+1)^{1/2}}{\sqrt{4}}\right| - \frac{2}{1}(x^2+1)^{1/2}$$

Thus, it has been that,

$$I = -2\sqrt{x^2+1} + \ln(x+\sqrt{x^2+1})$$

1.3..3When $a > 0$ e $\Delta > 0$

Let the expression be (57). This is,

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right).\ln\left(\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2\sqrt{a}}{\sqrt{\Delta}}(ax^2+bx+c)^{1/2}\right) + \frac{p}{a}(ax^2+bx+c)^{1/2} \quad (88)$$

As a> 0 and Δ> 0, the solution of the integral with m = 1/2 can be solved based on the expression given by (88).

Example1

Calculate the integral,

$$I = \int \frac{x+2}{\sqrt{x^2+2x-3}}dx$$

$$p = 1, q = 2, a = 1, b = 2 \text{ e } c = -3, \Delta = 16$$

As a> 0 and Δ> 0, the expression given by (84) is used.

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right).\ln\left(\frac{|2ax+b|}{\sqrt{\Delta}} + \frac{2\sqrt{a}}{\sqrt{\Delta}}(ax^2+bx+c)^{1/2}\right) + \frac{p}{a}(ax^2+bx+c)^{1/2}$$

Taking the given values, we obtain that,

$$I = \sqrt{x^2 + 2x - 3} + \ln\left[\frac{(x+1) + \sqrt{x^2 + 2x - 3}}{2}\right]$$

At where,

$$\frac{1}{\sqrt{a}}\left(q - \frac{Pb}{2a}\right) = 1$$

$$\frac{p}{a} = 1$$

$$\frac{|2ax + b|}{\sqrt{\Delta}} = \frac{x+1}{2}$$

$$\frac{2\sqrt{a}}{\sqrt{\Delta}}(ax^2 + bx + c)^{1/2} = \frac{1}{2}\sqrt{x^2 + 2x - 3}$$

1.3.3 When o $a > 0$e $\Delta = 0$

In this case we must prove a primitive for this particular case in which the quadratic function has a double root.

Demonstration of theorem

Be the

$$\text{integral} I = \int \frac{(px+q)dx}{\sqrt{ax^2+bx+c}} \tag{89}$$

Whereas

$$\sqrt{ax^2 + bx + c} = e^y \tag{90}$$

$$ax^2 + bx + c = e^{2y} \tag{91}$$

Deriving the expression (91), we have:

$$(2ax + b)dx = 2e^{2y}dy \tag{92}$$

Soon

$$dx = \frac{2e^{2y}dy}{2ax+b} \tag{93}$$

Taking (93) in (92), we obtain:

$$I = \int \frac{(px+q)}{e^y} . \frac{2.e^{2y}dy}{2ax+b} = 2. \int \frac{(Px+q)}{2ax+b} e^y dy \tag{94}$$

Let's get the variable x.
As

$$ax^2 + bx + c = e^{2y}$$

Leading to the first term, we obtain,

$$ax^2 + bx + (c - e^{2y}) = 0 \tag{95}$$

Assuming that the expression in (95) is a quadratic equation, we have the result of the discriminant Δ_1,,

$$\Delta_1 = b^2 - 4a.(c - e^{2y}) = b^2 - 4ac + 4ae^{2y} = \Delta + 4ae^{2y} \tag{96}$$

Since the roots are equal, we have Δ = 0. Soon. Whose roots assume the values,

$$x = \frac{-b \pm \sqrt{\Delta_1}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{\Delta + 4ae^{2y}}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ae^{2y}}}{2a}$$

Or

$$2ax + b = \pm 2\sqrt{a}e^y \tag{97}$$

Isolating x, we obtain,

$$x = \frac{-b}{2a} \pm \frac{\sqrt{a}e^y}{a} \tag{98}$$

For the function px + q given by the numerator of (94) and with the value given by (98), it follows that,

$$px + q = p.\left(\frac{-b}{2a} \pm \frac{\sqrt{a}e^y}{a}\right) + q \quad (99)$$

Whereas

$$v_x = \frac{-b}{2a} \quad (100)$$

As the vertex of the quadratic function. Taking (100) in (99), we obtain,

$$px + q = pv_x \pm p\frac{\sqrt{a}e^y}{a} + q \quad (101)$$

Taking expression (101) and taking in (94), we obtain, in view of expression (97), it follows that,

$$I = 2\int \frac{(px+q)}{2ax+b}e^y dy = I = 2\int \left[\left(pv_x \pm \frac{\sqrt{a}e^y}{a}P + q\right).\frac{e^y}{2\sqrt{a}e^y}\right].dy$$

$$I = \frac{1}{\sqrt{a}}\int \left[\left(pv_x \pm \frac{\sqrt{a}e^y}{a}p + q\right)\right].dy$$

$$I = \frac{1}{\sqrt{a}}\int \left[\left(pv_x \pm \frac{\sqrt{a}e^y}{a}p + q\right)\right].dy$$

$$I = \frac{1}{\sqrt{a}}\int \left[\left(\frac{-bp}{2a}dy \pm \frac{p\sqrt{a}}{a}e^y dy + qdy\right).\right]$$

$$I = \frac{1}{\sqrt{a}}\int \left[(\pm\frac{bp}{2a} \mp q)dy \pm \frac{p\sqrt{a}}{a}e^y dy\right]$$

Which leads to the following primitives,

$$I = \frac{1}{\sqrt{a}}\int \left[(\pm\frac{bp}{2a} \mp q)dy \pm \frac{p\sqrt{a}}{a}e^y dy\right]$$

$$I = \frac{1}{\sqrt{a}}\left(\pm\frac{bp}{2a} \mp q\right)y + \frac{p}{a}e^y \quad (102)$$

As

$$ax^2 + bx + c = e^{2y} \quad (103)$$

We have to,

$$ln(ax^2bx + c) = 2y \quad (104)$$

Or

$$y = \frac{1}{2}ln(ax^2bx + c) = ln\sqrt{ax^2 + bx + c} \quad (105)$$

$$\sqrt{ax^2 + bx + c} = e^y \quad (106)$$

We have, therefore,

$$I = \frac{1}{\sqrt{a}}\left(\pm\frac{bp}{2a} \mp q\right)ln\sqrt{ax^2 + bx + c} + \frac{p}{a}\sqrt{ax^2 + bx + c} \quad (107)$$

Denoting

$$k = \frac{1}{\sqrt{a}}\left(\pm\frac{bp}{2a} \mp q\right) \quad (108)$$

We obtain that,

$$I = k.ln\sqrt{ax^2 + bx + c} + \frac{p}{a}\sqrt{ax^2 + bx + c} \quad (109)$$

That is the integral sought. As for the signal of the coefficient k, we can consider for the first root of the quadratic function. This is

$$,k = \frac{1}{\sqrt{a}}\left(q - \frac{bp}{2a}\right) \quad (110)$$

Thus, we must,

$$I = \frac{1}{\sqrt{a}}\left(q - \frac{bp}{2a}\right).ln\sqrt{ax^2 + bx + c} + \frac{p}{a}\sqrt{ax^2 + bx + c} \quad (111)$$

Example 1

Let's calculate the Integral:

$$I = \int \frac{-3x+4}{\sqrt{4x^2-4x+1}}dx$$

Seja $p = -3, q = 4, a = 4, b = -4$ e $\Delta = 0$

soon,

$$\sqrt{ax^2+bx+c} = \sqrt{4x^2-4x+1} = \sqrt{(2x-1)^2} = 2x-1$$

being

$$K = -\frac{bp}{2a} + q \rightarrow K = \frac{5}{2}$$

Soon,

$$I = \frac{P}{a}\sqrt{(2x-1)^2} + \left(\frac{5}{2}\right).\frac{1}{\sqrt{4}} ln\sqrt{(2x-1)^2} \rightarrow$$

$$I = -\frac{3}{4}(2x-1) + \frac{5}{4}\ln(2x-1)$$

If we were to use the usual method, we would have:

$$I = \int \frac{(-3x+4)}{\sqrt{4x^2-4x+1}}dx = \int \frac{(-3x+4)}{2x-1}dx \rightarrow$$

$$2x-1 = u \rightarrow x = \frac{u+1}{2} \text{ e } dx = \frac{du}{2} \rightarrow$$

$$-3x+4 = -3.\left(\frac{u+1}{2}\right) + 4 = -\frac{3u}{2} - \frac{3}{2} + u = -\frac{3u}{2} + \frac{8-3}{2} = \frac{-3u}{2} + \frac{5}{2}$$

$$\rightarrow I = \int \left(-\frac{3u+5}{2}\right).\frac{du}{2}.\frac{1}{u} = \int \left(-\frac{3du}{4}\right) + \int \frac{5}{4}\frac{du}{u} \rightarrow$$

$$I = -\frac{3}{4}u + \frac{5}{4}lnu \rightarrow I = -\frac{3}{4}(2x-1) + \frac{5}{4}\ln(2x-1)$$

Which is in full agreement with the previous result.

Example 2

Calculate Integral

$$I = \int \frac{(2x-7)dx}{\sqrt{9x^2-6x+1}}$$

In this case, we must:

$$P = 2, q = -7, a = 9, b = -6 \text{ e}$$

$$\Delta = b^2 - 4ac = 36 - 4.9.(1) = 36 - 36 \rightarrow \Delta = 0$$

Wehaveto

$$K = \left(-\frac{bP}{2a} + q\right)\frac{1}{\sqrt{a}} \rightarrow$$

$$K = -\frac{19}{9}$$

$$I = \frac{P}{a}.(3x-1) - \frac{19}{9}.\ln(3x-1) \rightarrow I = \frac{2}{9}(3x-1) - \frac{19}{9}\ln(3x-1)$$

This represents the Integral sought.

2nd method: By changes of variables, Let's apply the usual method:

$$I = \int \frac{2x-7}{3x-1}dx \rightarrow 3x-1 = u \rightarrow$$

$$dx = \frac{du}{3} \rightarrow I = \int \frac{(2x-7)}{u}.\frac{du}{3} = \int \frac{1}{3u}\left[2.\left(\frac{u+1}{3}\right) - 7\right]du$$

$$I = \int \frac{1}{3u}\left(\frac{2u+2-21}{3}\right)du = \int \frac{1}{3u}\left(\frac{2u-19}{3}\right)du \rightarrow$$

$$I = \frac{2}{9}\int du - \frac{19}{9}\int \frac{du}{u} = \frac{2}{9} - \frac{19}{9}lnu \rightarrow$$

$$I = \frac{2}{9}(3x-1) - \frac{19}{9}\ln(3x-1)$$

Which fully agrees with the previous method.

II. Conclusion

We verify from the previous text that the insertion of the alternative method can replace the traditional method that happens through changes of variables. In this case, for integrals studied for m = 1/2, it becomes a less complicated mathematical development, in view of the direct substitution of the parameters in the integral given in the literal expression. We observe from the exercises that there is a consistency with an understandable methodology where the student does not resort to changing variables or other techniques to obtain a particular primitive. In this case, it is enough to understand the different cases to be used and to apply the values of the parameters in a specific integral by performing the substitution of the parameters in the integral.

The restlessness might arise in considering why we substitute a quadratic function for the equality of an exponential function. In this case, we can eliminate this doubt by considering that the exponential is the only function in mathematics that remains unchanged at less than one constant. In this case, it can be used to calculate integrals by replacing other functions present in the integrand, because at the end of the result it is possible to find the primitive, as it was done at the beginning of the text to obtain the fundamental integral.

References

[1] DOWLING, E. T. Introduction to Mathematical Economics. 3rd ed. New York: McGraw-Hill, 2001. Schaum`s Outline Series.

[2] BALDANI, J.; BRADFIELD, J. & TURNER, R.W. Mathematical Economics. 2nd ed. Chulo Alto, CA: Southwestern College Publications, 2004.

[3] SIMON, C.P. & BLUME, L. Mathematics for Economists. New York: W.W. Norton & Company Inc., 1994.

[4] J. Stewart, Calculus, Brooks/Cole Publ. Co., 1999.

[5] J. Stewart, Calculus, Brooks/Cole Publ. Co., 1999. P. Z. Taboas, **Calculation in a Real Variable**, ICMC-USP, 2003. (Disponívelnagráfica)

[6] TALL, D. Students' Difficulties in Calculus. In: INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION - ICME, Plenary presentation in Working Group 3. Québec, Anais... Québec, CA, August 1992

[7] TARMIZI, R. A. Visualizing Students' Difficulties in Learning Calculus. In: INTERNATIONAL CONFERENCE ON MATHEMATICS EDUCATION RESEARCH 2010 (ICMER 2010). Procedia Social and Behavioral Sciences 8, p. 377-383, 2010.

[8] J.L.E. Dreyder. A History of Astronomy from Thales to Kepler, Dover Publications, New York, (1953).

[9] GRANT, E., Physical Science in the Middle Ages, John Wiley & Sons, New York, 1971.

[10] Imke de Pater and Jack J. Lissauer, Planetary Sciences (Cambridge University Press, Cambridge, 2001).

[11] Judith V. Grabiner; The Changing Concept of Change: the Derivative from Fermat to Weierstrass, Mathematics Magazine, vol.56, no.4, September 1983

[12] BARON, M. E., The Origins of the Infinitesimal Calculus, New York: Dover Publications, 1969.

[13] LOPEZ, R. J.,Maple via Calculus: a tutorial approach. Birkhauser: Berlin, 1994.

[14] RASMUSSEN, C.; MARRONGELLE, K.; BORBA, M. Research on calculus: what do we know and where do we need to go? ZDM MathematicsEducation, v. 46, p. 507–515, 2014.