On Square Degree Distance Of Mycielskian Graphs

Shailaja Shirakol[1]∗ , Manjula Kalyanshetti[2]∗∗

∗Department of Mathematics, SDMCET, Dharwad, Karnataka, India

∗∗Department of Mathematics, Jain College of Engineering, Belagavi, Karnataka, India

ABSTRACT:- The Square Degree Distance(SDD) a connected simple graph is defined as

$$SDD(G) = \sum_{\{u,v\}\subseteq V(G)} \text{mod} * -4mmd_G(u,v)^2[deg_G(u) + deg_G(v)],$$

Where $d_G(u,v)$ denotes the distance between the vertices v and u in G and $deg_G(u)$ is the degree of vertex u in G. In this paper we introduce SDD(G) of a molecular graph G and determine the exact value of the degree distance of Mycielskian graph with diameter. Also we determine exact value of degree distance of the complement of arbitrary Mycielskian graphs.

I. Introduction

Let $G = (V,E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. The complement of G, denoted by $\bar{G}$, is a graph which has the same vertices as G, and in which two vertices are adjacent if and only if they are not adjacent in G. The degree of a vertex v, denoted by $deg_G(v)$. The distance between two vertices of a graph is the number of edges in a shortest path connecting them it is denoted by $d_G(u,v)$. For undefined terminologies we refer the reader to [11].

A graph invariant is any function on a graph that does not depend on a labeling of its vertices. Such quantities are also called topological indices. Hundreds of different invariants have been employed to date (with unequal success) in QSAR/QSPR studies. Among more useful of them appear two that are known under various names, but mostly as Zagreb indices. Due to their chemical relevance they have been subject of numerous papers in chemical literature [5,6,8,10]. There are two invariants called the first Zagreb index and second Zagreb index [3,9,12,14], defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \text{mod}10mmand\text{mod}10mmM_2(G) = \sum_{uv\in E(G)} d_G(u)\, d_G(v),$$

respectively.

The Wiener index is named after Harry Wiener, who introduced it in 1947; at the time, Wiener called it the "path number"[16]. It is the oldest topological index related to molecular branching[15]. Based on its success, many other topological indexes of chemical graphs, based on information in the distance matrix of the graph, have been developed subsequently to Wiener's work.

Definition 1. let G be any connected graph of order n and size m. Then Wiener index of G is denoted by $W(G)$ and is defined as follows.

$$W(G) = \frac{1}{2}\sum_{u\in V(G)} d_G(u,v)$$

Motivated by the first Zagreb index we can define the sum of square of the distance as

$$W^*(G) = \frac{1}{2}\sum_{u\in V(G)} d_G(u,v)^2$$

The square degree distance of G, denoted by $SDD(G)$, is defined as follows

$$SDD(G) = \sum_{\{u,v\}\subseteq V(G)} \text{mod} * -4mmd_G(u,v)^2[deg_G(u) + deg_G(v)].$$

Definition 2. For a graph $G = (V,E)$, the Mycielskian of G is the graph $\mu(G)$(or simply μ) is defined as the

graph having vertex set $V \cup X \cup \{x\}$ and edge set $E \cup \{v_i x_j : v_i v_j \in E\} \cup \{x_i x_j : 1 \leq j \leq n\}$, where $V = \{v_1, v_2, \cdots, v_n\}$ and $X = \{x_1, x_2, \cdots, x_n\}$.
For more details on Mycielskian graph see [1, 2, 13]. In this paper we determine the exact value of the degree distance of Mycielskian graph with diameter. Also we determine exact value of degree distance of the complement of arbitrary Mycielskian graphs.

II. Results

We begin with the following straightforward, previously known, auxiliary result.

Observation 1[2] *Let μ be the Mycielskian of G. Then for each $v \in V(\mu)$ we have*

$$deg_\mu(v) = \begin{cases} n, & v = x; \\ 1 + deg_G(v_i), & v = x_i; \\ 2deg_G(v_i), & v = v_i. \end{cases}$$

Observation[2]**2** *In the Mycielskian μ of G, the distance between two vertices $u, v \in V(\mu)$ are given as follows*

$$d_\mu(u,v) = \begin{cases} 1 & u = x, v = x_i \\ 2 & u = x, v = v_i \\ 2 & u = x_i, v = x_j \\ d_G(v_i, v_j) & u = v_i, v = v_j, d_G(v_i, v_j) \leq 3 \\ 4 & u = v_i, v = v_j, d_G(v_i, v_j) \geq 4 \\ 2 & u = v_i, v = x_j, i = j \\ d_G(v_i, v_j) & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) \leq 2 \\ 3 & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) \geq 3 \end{cases}$$

Specially, the diameter of the Mycielskian graph is at most four.

Lemma[2]**3** *Let G be a graph of size m whose vertex set is $V = \{v_1, v_2, \cdots, v_n\}$. Then*

$$\sum_{\{u,v\} \subseteq V(G)} \text{mod} * -4mm[deg_G(u) + deg_G(v)] = (n-1)2m$$

Observation [2]**4** *Let $\overline{\mu}$ be the complement of Mycielskian of G. Then for each $v \in V(\overline{\mu})$ we have*

$$deg_{\overline{\mu}}(v) = \begin{cases} n, & v = x; \\ 2n - (1 + deg_G(v_i)), & v = x_i; \\ 2n - 2deg_G(v_i), & v = v_i. \end{cases}$$

Observation [2]**5** *In the complement of Mycielskian μ of G, the distance between two vertices $u, v \in V(\overline{\mu})$ are given as follows*

$$d_{\overline{\mu}}(u,v) = \begin{cases} 2 & u = x, v = x_i \\ 1 & u = x, v = v_i \\ 1 & u = x_i, v = x_j \\ 1 & u = v_i, v = v_j, d_G(v_i, v_j) > 1 \\ 2 & u = v_i, v = v_j, d_G(v_i, v_j) = 1 \\ 1 & u = v_i, v = x_j, i = j \\ 1 & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) > 1 \\ 2 & u = v_i, v = x_j, i \neq j, d_G(v_i, v_j) = 1 \end{cases}$$

Specially, the diameter of $\overline{\mu}$ is exactly 2.

Now, we are in a position to establish the exact value of the degree distance of Mycielskian graph $\mu(G)$ of a graph G.

Theorem 6 *Let G be a (n,m)-graph with $diam(G) = 2$. If μ is the Mycielskian of G, then the distance degree sum of μ is given by*
$SDD(\mu) = 5SDD(G) + W^*(G) + 4(n^2 + 2n + 7m - 1) + n(n+1) + 2m.$

Proof. We consider the following different possible cases.
Case 1. $u = x$ and $v \in X$.

$$\sum_{i=1}^{n} d_\mu(x,x_i)^2[deg_\mu(x) + deg_\mu(x_i)] = \sum_{i=1}^{n} 1^2[n + 1 + deg_G(v_i)]$$
$$= n(n+1) + 2m.$$

Case 2. $u = x$ and $v \in V$.

$$\sum_{i=1}^{n} d_\mu(x,v_i)^2[deg_\mu(x) + deg_\mu(v_i)] = \sum_{i=1}^{n} 4[n + 2deg_G(v_i)]$$
$$= 4(n^2 + 2m)$$

Case 3. $\{u,v\} \subseteq X$.
Using Lemma 1, we have

$$\sum_{\{x_i,x_j\}\subseteq X} d_\mu(x_i,x_j)^2[deg_\mu(x_i) + deg_\mu(x_j)] = \sum_{\{x_i,x_j\}\subseteq X} 4[1 + deg_G(v_i) + 1 + deg_G(v_j)]$$
$$= \frac{8n(n-1)}{2} + \sum_{\{x_i,x_j\}\subseteq X} [deg_G(v_i) + deg_G(v_j)]$$
$$= 4n(n-1) + 4m(n-1)$$
$$= 4(n-1)(n+m).$$

Case 4. $\{u,v\} \subseteq V$.
Since the diameter of G is two, Observation 2 implies that $d_\mu(v_i,v_j) = d_G(v_i,v_j)$. Hence

$$\sum_{\{v_i,v_j\}\subseteq V} d_\mu(v_i,v_j)^2[deg_\mu(v_i) + deg_\mu(v_j)] = \sum_{\{v_i,v_j\}\subseteq V} d_G(v_i,v_j)^2[2deg_G(v_i) + 2deg_G(v_j)]$$
$$= 2SDD(G)$$

Case 5. $u = v_i$ and $v = x_i$, $1 \le i \le n$

$$\sum_{i=1}^{n} d_\mu(v_i,x_i)^2[deg_\mu(v_i) + deg_\mu(x_i)] = \sum_{i=1}^{n} 4[2deg_G(v_i) + 1 + deg_G(v_i)]$$
$$= 20m + 4n$$
$$= 4(n + 5m).$$

Case 6. $u = v_i$ and $v = x_j$, $i \neq j$

$$\sum_{\{v_i x_j\}\subseteq V(\mu)_{i\neq j}} \mathrm{mod} * -9mmd_\mu(v_i,x_j)^2[deg_\mu(v_i) + deg_\mu(x_j)] = \mathrm{mod} * -9mm \sum_{\{v_i x_j\}\subseteq V(\mu)_{i\neq j}} \mathrm{mod} * -7mmd_\mu(v_i,x_j)^2[2deg_G(v_i) + 1 + deg_G(v_j)]$$
$$= 2\mathrm{mod} * -7mm \sum_{\{v_i x_j\}\subseteq V(\mu)_{i\neq j}} \mathrm{mod} * -9mmd_\mu(v_i,x_j)^2 deg_G(v_i) + W^*(\mu) + \mathrm{mod} * -7mm \sum_{\{v_i x_j\}\subseteq V(\mu)_{i\neq j}} \mathrm{mod} * -9mmd_\mu(v_i,x_j)^2 deg_G(v_j)$$

Since $d_\mu(v_i,x_j)^2 = d_\mu(v_j,x_i)^2$, $d_G(x_i,v_i)^2 = 0$, using Observation2, we get

$$\sum_{\{v_i x_j\}\subseteq V(\mu)_{i\neq j}} \mathrm{mod} * -9mmd_\mu(v_i,x_j)^2[deg_\mu(v_i) + deg_\mu(x_j)] = \mathrm{mod} * -9mm \sum_{\{v_i x_j\}\subseteq V(\mu)_{i\neq j}} \mathrm{mod} * -7mmd_\mu(v_i,v_j)^2 deg_G(v_i)$$
$$= \sum_{\{v_i,x_j\}\subseteq V(\mu)} d_G(v_i,v_j)^2 deg_G(v_i)$$
$$= \sum_{\{i,j\}\subseteq V} d_G(v_i,v_j)^2[deg_G[v_i + deg_G(v_j)]]$$
$$= SDD(G).$$

Hence

$$\sum_{\{v_i,x_j\}\subseteq V(\mu)_{i\neq j}} \text{mod} * -9mmd_{\mu}(v_i,x_j)^2[deg_{\mu}(v_i)+deg_{\mu}(x_j)] = 2SDD(G)+W^*(\mu)+SDD(G)$$
$$= 3SDD(G)+W^*(\mu).$$

Thus, the result follows by combining these cases.

Theorem 7 *Let G be a (n,m)-graph and let $\overline{\mu}$ be the complement of the Mycielskian μ of G. Then the degree distance of $\overline{\mu}$ is given by*

$$SDD(\overline{\mu}) = 11n^2-12m+(n-1)(4n^2-10m-n)+n(4n-1)+n(n-1)(4n-1)+6m(3n-1) - 3M_1(G).$$

Proof. By using the definition of degree distance of a graph G, we have the following possible cases.

Case 1. $u = x$ and $v \in X$

$$\sum_{i=1}^{n} d_{\overline{\mu}}(x_i,x_j)^2[deg_{\overline{\mu}}(x_i)+deg_{\overline{\mu}}(x_j)] = \sum_{i=1}^{n} 2^2[n+2n-n-deg_G(v_i)]$$
$$= 2(4n^2-m).$$

Case 2. $u = x$ and $v \in V$

$$\sum_{i=1}^{n} d_{\overline{\mu}}(x_i,v_i)^2[deg_{\overline{\mu}}(x_i)+deg_{\overline{\mu}}(v_i)] = \sum_{i=1}^{n} 1^2[n+2n-2deg_G(v_i)]$$
$$= 3n^2-4m.$$

Case 3. $\{u,v\}\subseteq X$.
Using Lemma 1, we get

$$\sum_{\{x_i,x_j\}\subseteq X} d_{\overline{\mu}}(x_i,x_j)^2[deg_{\overline{\mu}}(x_i)+deg_{\overline{\mu}}(x_j)] = \sum_{\{x_i,x_j\}\subseteq X} 1^2[2n-1-deg_G(v_i)+2n-1-deg_G(v_j)]$$
$$= (4n-2)(\frac{n(n-1)}{2})+\sum_{\{v_i,v_j\}\subseteq V}[deg_G(v_i)+deg_G(v_j)]$$
$$= n(n-1)(2n-1)+2m(n-1).$$

Case 4. $\{u,v\}\subseteq V$.
By Observation 4, $d_{\overline{\mu}}(v_i,v_j)^2 = 1$ whenever $v_iv_j \notin E(G)$ and is 2 otherwise. Also
$\{\{v_i,v_j\}\subseteq V: i\neq j, v_iv_j\notin E(G)\} = \{\{v_i,v_j\}\subseteq V: i\neq j\}\backslash\{\{v_i,v_j\}\subseteq V: v_iv_j\in E(G)\}$.
Thus,

$$\sum_{\{v_i,v_j\}\subseteq V} d_{\overline{\mu}}(v_i,v_j)^2[deg_{\overline{\mu}}(v_i)+deg_{\overline{\mu}}(v_j)] = \sum_{v_iv_j\notin E} 1^2[2n-2deg_G(v_i)+2n-2deg_G(v_j)]$$
$$+ \sum_{v_iv_j\in E} 2[2n-2deg_G(v_i)+2n-2deg_G(v_j)]$$
$$= 4n(\frac{n(n-1)}{2})-4m(n-1)+8mn-8m(n-1)$$
$$= 2n^2(n-1)-12m(n-1)+8mn.$$

Case 5. $u = v_i$ and $v = x_i; 1\leq i\leq n$.

$$\sum_{i=1}^{n} d_{\overline{\mu}}(v_i,x_i)^2[deg_{\overline{\mu}}(v_i)+deg_{\overline{\mu}}(x_i)] = \sum_{i=1}^{n} 1^2[2n-2deg_G(v_i+2n-1-deg_G(v_i))]$$
$$= n(4n-1)-6m.$$

Case 6. $u = v_i$ and $v = x_j; i \neq n$.
By Observation 4, $d_{\bar{\mu}}(v_i, x_j)^2 = d_{\bar{\mu}}(v_j, x_i)^2 = 1$ whenever $v_i v_j \notin E(G)$ and is 2 otherwise. Also
$\{(v_i, v_j): i \neq j, v_i v_j \notin E(G)\} = \{(v_i, v_j): i \neq j\} \backslash \{(v_i, v_j): v_i v_j \in E(G)\}$.
Thus,

$$\sum_{\substack{\{v_i,x_j\}\subseteq V(\bar{\mu})\\ i\neq j}} \mathrm{mod} * -9mm d_{\bar{\mu}}(v_i,x_j)^2[deg_{\bar{\mu}}(v_i)+deg_{\bar{\mu}}(x_j)] = \sum_{\substack{(v_i,v_j)\\ v_iv_j\notin E(G)}} \mathrm{mod} * -9mm 1^2[2n-2deg_G(v_i)+2n-1-deg_G(v_j)]$$

$$+ \sum_{\substack{(v_i,v_j)\\ v_iv_j\in E(G)}} \mathrm{mod} * -9mm 2[2n-2deg_G(v_i)+2n-1-deg_G(v_j)]$$

Each vertex v_j can be paired with n-1 vertices v_i as $(v_i, v_j), i \neq j$. Hence $\sum_{(v_i,v_j)} deg_G(v_j) = (n-1)\sum_{j=1}^{n} deg_G(v_j)$ which is equal to $(n-1)2m$. Also note that $|\{(v_i, v_j): i \neq j\}| = n(n-1)$ $|\{v_i, v_j\}: v_i v_j \in E(G)| = 2m$ and $\sum_{\substack{(v_i,v_j)\\ v_iv_j\in E(G)}} \mathrm{mod} * -9mm (deg_G(v_i))^2$. We obtain

$$\sum_{\substack{\{v_i,x_j\}\subseteq V(\bar{\mu})\\ i\neq j}} \mathrm{mod} * -9mm d_{\bar{\mu}}(v_i,x_j)^2[deg_{\bar{\mu}}(v_i)+deg_{\bar{\mu}}(x_j)] = n(n-1)(4n-1)+6m(3n-1)-3M_1(G).$$

Thus, the result follows by combining these cases.

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